29 Chapter: Symmetry and Counting

A group G is said to *act on* a set S if there is a homomorphism from G to $\mathrm{sym}(S)$ where $\mathrm{sym}(S)$ is the group of all permutations on S. In this chapter we will consider the action of G on itself given by conjugation. That is, given $g \in G$, define $\phi_g : G \to G$ by $\phi_g(h) = ghg^{-1}$ for all $h \in G$. Then ϕ_g is a permutation of the elements in G (that is, $\phi_g \in \mathrm{sym}(G)$). The map $\Phi : G \to \mathrm{sym}(G)$ given by $\Phi(g) = \phi_g$ is a homomorphism. Thus this map Φ gives an action of G on itself.

For elements a, b in a group G the command `a^b` in GAP computes $b^{-1}ab$. For example:

```
gap> (2,3,4)^(1,4,2);
(1,3,2)
gap> (1,2,4)*(2,3,4)*(1,4,2);
(1,3,2)
```

Note: Remember GAP multiplies permutations from left to right.

Given an action, Φ, of a group G on a set S, define the *kernel of the action* to be the set $\{g \in G \mid \Phi(g) = e\}$ where e denotes the identity in $\mathrm{sym}(S)$. The question we will investigate is: What is the kernel of the action when G acts on itself by conjugation? First we will consider this question when G is a cyclic group of order 4.

```
gap> G:=Group((1,2,3,4));
Group([ (1,2,3,4) ])
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3,4)],[(1,2,3,4)^()]);
[ (1,2,3,4) ] -> [ (1,2,3,4) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3,4)],[(1,2,3,4)^(1,2,3,4)]);
[ (1,2,3,4) ] -> [ (1,2,3,4) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3,4)],[(1,2,3,4)^(1,3)(2,4)]);
[ (1,2,3,4) ] -> [ (1,2,3,4) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3,4)],[(1,2,3,4)^(1,4,3,2)]);
[ (1,2,3,4) ] -> [ (1,2,3,4) ]
```

Since conjugation by any element in G maps the generator $(1,2,3,4)$ back to $(1,2,3,4)$, we see the kernel of the action is all of G.

Exercises

29.1 Prove that the maps ϕ_g and Φ defined above are homomorphisms.

29.2 Prove G is Abelian if and only if the kernel of the action of G on itself by conjugation is G.

In general, $g \in G$ will be in the kernel if and only if ϕ_g maps each element of a set of generators of G to itself. The below output investigates the kernel of this conjugation action of S_3 on itself.

```
gap> G:=SymmetricGroup(3);
Sym( [ 1 .. 3 ] )
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(),(1,2)^()]);
[ (1,2,3), (1,2) ] -> [ (1,2,3), (1,2) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(2,3),
> (1,2)^(2,3)]);
[ (1,2,3), (1,2) ] -> [ (1,3,2), (1,3) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(1,2),
> (1,2)^(1,2)]);
[ (1,2,3), (1,2) ] -> [ (1,3,2), (1,2) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(1,3),
> (1,2)^(1,3)]);
[ (1,2,3), (1,2) ] -> [ (1,3,2), (2,3) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(1,2,3),
> (1,2)^(1,2,3)]);
[ (1,2,3), (1,2) ] -> [ (1,2,3), (2,3) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(1,3,2),
> (1,2)^(1,3,2)]);
[ (1,2,3), (1,2) ] -> [ (1,2,3), (1,3) ]
```

Thus we see the kernel of this action contains only the identity of S_3.

Exercises

29.3 Assume G is a group that is generated by two elements. Write a subroutine in GAP that lists the elements in the kernel of the action of a group on itself by conjugation.

29.4 Use your subroutine in Exercise 29.3 to find the kernel of the conjugation action when G is S_6, D_{12}, D_{19} and A_4.

29.5 For any group G the kernel of the conjugation action is a familiar subgroup. Use your answers to Exercise 29.4 to help you conjecture what the kernel is in general.

29.6 Prove your conjecture in Exercise 29.5

