

## 29 Chapter: Symmetry and Counting

A group  $G$  is said to *act on* a set  $S$  if there is a homomorphism from  $G$  to  $\text{sym}(S)$  where  $\text{sym}(S)$  is the group of all permutations on  $S$ . In this chapter we will consider the action of  $G$  on itself given by conjugation. That is, given  $g \in G$ , define  $\phi_g : G \rightarrow G$  by  $\phi_g(h) = ghg^{-1}$  for all  $h \in G$ . Then  $\phi_g$  is a permutation of the elements in  $G$  (that is,  $\phi_g \in \text{sym}(G)$ ). The map  $\Phi : G \rightarrow \text{sym}(G)$  given by  $\Phi(g) = \phi_g$  is a homomorphism. Thus this map  $\Phi$  gives an action of  $G$  on itself.

For elements  $a, b$  in a group  $G$  the command `a^b` in GAP computes  $b^{-1}ab$ . For example:

```
gap> (2,3,4)^(1,4,2);
(1,3,2)
gap> (1,2,4)*(2,3,4)*(1,4,2);
(1,3,2)
```

*Note:* Remember GAP multiplies permutations from left to right.

Given an action,  $\Phi$ , of a group  $G$  on a set  $S$ , define the *kernel of the action* to be the set  $\{g \in G \mid \Phi(g) = e\}$  where  $e$  denotes the identity in  $\text{sym}(S)$ . The question we will investigate is: What is the kernel of the action when  $G$  acts on itself by conjugation? First we will consider this question when  $G$  is a cyclic group of order 4.

```
gap> G:=Group((1,2,3,4));
Group([ (1,2,3,4) ])
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3,4)],[(1,2,3,4)^()]);
[ (1,2,3,4) ] -> [ (1,2,3,4) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3,4)],[(1,2,3,4)^(1,2,3,4)]);
[ (1,2,3,4) ] -> [ (1,2,3,4) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3,4)],[(1,2,3,4)^(1,3)(2,4)]);
[ (1,2,3,4) ] -> [ (1,2,3,4) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3,4)],[(1,2,3,4)^(1,4,3,2)]);
[ (1,2,3,4) ] -> [ (1,2,3,4) ]
```

Since conjugation by any element in  $G$  maps the generator  $(1, 2, 3, 4)$  back to  $(1, 2, 3, 4)$ , we see the kernel of the action is all of  $G$ .

### *Exercises*

29.1 Prove that the maps  $\phi_g$  and  $\Phi$  defined above are homomorphisms.

29.2 Prove  $G$  is Abelian if and only if the kernel of the action of  $G$  on itself by conjugation is  $G$ .

In general,  $g \in G$  will be in the kernel if and only if  $\phi_g$  maps each element of a set of generators of  $G$  to itself. The below output investigates the kernel of this conjugation action of  $S_3$  on itself.

```

gap> G:=SymmetricGroup(3);
Sym( [ 1 .. 3 ] )
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(),(1,2)^()]);
[ (1,2,3), (1,2) ] -> [ (1,2,3), (1,2) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(2,3),
> (1,2)^(2,3)]);
[ (1,2,3), (1,2) ] -> [ (1,3,2), (1,3) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(1,2),
> (1,2)^(1,2)]);
[ (1,2,3), (1,2) ] -> [ (1,3,2), (1,2) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(1,3),
> (1,2)^(1,3)]);
[ (1,2,3), (1,2) ] -> [ (1,3,2), (2,3) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(1,2,3),
> (1,2)^(1,2,3)]);
[ (1,2,3), (1,2) ] -> [ (1,2,3), (2,3) ]
gap> f:=GroupHomomorphismByImages(G,G,[(1,2,3),(1,2)],[(1,2,3)^(1,3,2),
> (1,2)^(1,3,2)]);
[ (1,2,3), (1,2) ] -> [ (1,2,3), (1,3) ]

```

Thus we see the kernel of this action contains only the identity of  $S_3$ .

### *Exercises*

29.3 Assume  $G$  is a group that is generated by two elements. Write a subroutine in **GAP** that lists the elements in the kernel of the action of a group on itself by conjugation.

29.4 Use your subroutine in Exercise 29.3 to find the kernel of the conjugation action when  $G$  is  $S_6$ ,  $D_{12}$ ,  $D_{19}$  and  $A_4$ .

29.5 For any group  $G$  the kernel of the conjugation action is a familiar subgroup. Use your answers to Exercise 29.4 to help you conjecture what the kernel is in general.

29.6 Prove your conjecture in Exercise 29.5