27 Chapter: Symmetry Groups

The first figure on the page of figures at the end of this chapter is a 3-prism. The front and back faces are equilateral triangles. We will use GAP to help us show the group of rotations in $\mathbf{R}^3$ of a 3-prism is isomorphic to D_3. [Gallian, Chapter 27, Exercise 4] Let G denote this group of rotations. Label the vertices of the facing triangle 1,2 and 3. Label the vertices of the other triangle in the prism 4,5 and 6. (See figure of the labeled 3-prism on last page of this chapter.) The group of rotations must be a subgroup of the group of permutations of the set $\{1,2,3,4,5,6\}$. There are two types of rotations. We can rotate each triangle the same amount. Thus $(1,2,3)(4,5,6)$ is in G. We can also rotate the front facing triangle to the back. Thus the rotation $(1,4)(2,6)(3,5)$ is in G.

```
gap> G:=Subgroup(SymmetricGroup(6), [(1,2,3)(4,5,6), (1,4)(2,6)(3,5)]);
Group([ (1,2,3)(4,5,6), (1,4)(2,6)(3,5) ])
gap> Elements(G);
[ (), (1,2,3)(4,5,6), (1,3,2)(4,6,5), (1,4)(2,6)(3,5), (1,5)(2,4)(3,6),
  (1,6)(2,5)(3,4) ]
```

The above exhibits G as a subgroup of S_6. We can now have GAP set up an isomorphism between D_3 and G. (We use here that $D_3 \cong S_3$.)

```
gap> d3:= SymmetricGroup(3);
Sym( [ 1 .. 3 ] )
gap> IsomorphismGroups(d3,G);
[ (2,3), (1,2,3) ] -> [ (1,4)(2,6)(3,5), (1,2,3)(4,5,6) ]
```

That is, the homomorphism that maps the generators $(2,3)$ and $(1,2,3)$ of D_3 to the generators $(1,4)(2,6)(3,5)$ and $(1,2,3)(4,5,6)$ respectively of G is an isomorphism. If a pair of groups are not isomorphic then this command returns `fail`:

```
gap> IsomorphismGroups(SymmetricGroup(3), AlternatingGroup(4));
fail
```

Exercises
For the exercises in this chapter see figures on the last page of this chapter.

27.1 Exhibit the group of rotations in $\mathbf{R}^3$ of a 4-prism as a subgroup of S_8. This group is isomorphic to which familiar group?

27.2 Exhibit the group of rotations in $\mathbf{R}^3$ of a 5-prism as a subgroup of S_{10}. This group is isomorphic to which familiar group?

27.3 Exhibit the group of rotations in $\mathbf{R}^3$ of a 6-prism as a subgroup of S_{12}. This group is isomorphic to which familiar group?

27.4 Make a conjecture about what the group of rotations in $\mathbf{R}^3$ of a n-prism is.

27.5 Prove your conjecture in Exercise 27.4.

27.6 The order of the symmetry group (including both rotations and reflections) in $\mathbf{R}^3$ of a 3-prism is 12. Exhibit this symmetry group as a subgroup of S_6. This group is isomorphic to which familiar group of order 12?

27.7 Exhibit the symmetry group in $\mathbf{R}^3$ of a 5-prism as a subgroup of S_{10}. This group is isomorphic to which familiar group?

27.8 Make a conjecture about what the symmetry group in $\mathbf{R}^3$ of an n-prism is.

27.9 Test your conjecture in Exercise 27.8 for $n = 4$.

Chapter 27 Figures