24 Chapter: Sylow Theorems

Let G be a finite group and let p be a prime that divides the order of G. Let p^k be the largest power of p that divides the order of G. A subgroup of G of order p^k is called a *Sylow p-subgroup* of G.

To do the exercises in this chapter you will need to fetch the file "sylows" from the web site. This file contains a function that returns a list of all the Sylow p-subgroups of a group for a given group and a given prime.

```
gap> Read("sylows");
gap> G:=SymmetricGroup(6);
Sym( [ 1 ..  6 ] )
gap> sylows(G,3);
The Sylow 3-subgroups of SymmetricGroup( [ 1 ..  6 ] ) are:
[ Group([ (1,2,3), (4,5,6) ]), Group([ (1,2,4), (3,5,6) ]),
Group([ (1,2,5), (3,4,6) ]), Group([ (1,2,6), (3,4,5) ]),
Group([ (1,3,4), (2,5,6) ]), Group([ (1,3,5), (2,4,6) ]),
Group([ (1,3,6), (2,4,5) ]), Group([ (1,4,5), (2,3,6) ]),
Group([ (1,4,6), (2,3,5) ]), Group([ (1,5,6), (2,3,4) ]) ]
```

From the above output we see that S_6 has ten Sylow 3-subgroups. The first Sylow 3-subgroup in the list is the subgroup of S_6 generated by $(1,2,3)$ and $(4,5,6)$. Observe that all ten Sylow 3-subgroups are generated by two disjoint 3-cycles. Thus the Sylow 3-subgroups are Abelian because disjoint cycles commute.

Exercises

24.1 **By hand** find all the Sylow p-subgroups of S_4 for every prime p that divides the order of S_4.

24.2 Use GAP to check your answer to Exercise 24.1.

24.3 Use GAP to find the **number** of Sylow p-subgroups in A_6 for each prime p that divides $|A_6|$. (Recall the command for the alternating group is `AlternatingGroup(n);`.)

24.4 Repeat Exercise 24.3 for the group S_7.

24.5 Repeat Exercise 24.3 for a cyclic group of order 60.

24.6 Make a conjecture about the number of Sylow p-subgroups of a group mod p.

