

## 19 Chapter: Vector Spaces

Recall from Chapter 13 that  $\mathbf{Z}_p$  is a field for every prime  $p$ . Let  $GF(p)^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbf{Z}_p\}$ . Then  $GF(p)^n$  is an  $n$  dimensional vector space over  $\mathbf{Z}_p$ . In this chapter we will investigate subspaces of  $GF(p)^n$ . For example consider the vector space  $GF(3)^2$ :

```
gap> V:=GF(3)^2;
( GF(3)^2 )
gap> Elements(V);
[ [ 0*Z(3), 0*Z(3) ], [ 0*Z(3), Z(3)^0 ], [ 0*Z(3), Z(3) ],
[ Z(3)^0, 0*Z(3) ], [ Z(3)^0, Z(3)^0 ], [ Z(3)^0, Z(3) ], [ Z(3), 0*Z(3) ],
[ Z(3), Z(3)^0 ], [ Z(3), Z(3) ] ]
```

(Recall a generator of the multiplicative group of units in  $\mathbf{Z}_p$  is denoted in GAP by  $Z(p)$ .)

*Careful:* Note that  $GF(p)^n$  is the direct product of  $n$  copies of  $\mathbf{Z}_p$  not the field of order  $p^n$ .

The vector space  $V$  is small enough that we can easily find all the 1-dimensional subspaces by hand. The `Display` command is useful here:

```
gap> Display(Elements(V));
. .
. 1
. 2
1 .
1 1
1 2
2 .
2 1
2 2
```

The zero is denoted by a dot and  $Z(3)$  is denoted by 2. The following GAP work shows the 1-dimensional subspaces:

```
gap> D:=Subspaces(V,1);
Subspaces( ( GF(3)^2 ), 1 )
gap> e:= Elements(D);;
gap> Display(Elements(e));
[ VectorSpace( GF(3), [ [ 0*Z(3), Z(3)^0 ] ] ),
  VectorSpace( GF(3), [ [ Z(3)^0, 0*Z(3) ] ] ),
  VectorSpace( GF(3), [ [ Z(3)^0, Z(3)^0 ] ] ),
  VectorSpace( GF(3), [ [ Z(3)^0, Z(3) ] ] ) ]
gap> Display(Elements(e[1]));
. .
. 1
. 2
```

```

gap> Display(Elements(e[2]));
. .
1 .
2 .
gap> Display(Elements(e[3]));
. .
1 1
2 2
gap> Display(Elements(e[4]));
. .
1 2
2 1

```

Thus, as expected, we see that the 1-dimensional subspaces are  $\{(0, 0), (0, 1), (0, 2)\}$ ,  $\{(0, 0), (1, 0), (2, 0)\}$ ,  $\{(0, 0), (1, 1), (2, 2)\}$ , and  $\{(0, 0), (1, 2), (2, 1)\}$ . If you just need to find the number of 1-dimensional subspaces you can type:

```

gap> Size(Subspaces(V,1));
4

```

### *Exercises*

19.1 **By hand** find all the 1-dimensional subspaces of  $GF(3)^3$ .

19.2 Use **GAP** to check your answer to Exercise 19.1.

19.3 Use **GAP** to find the number of 1-dimensional subspaces of  $GF(p)^3$  for  $p = 2, 5, 7$  and  $11$ .

19.4 Make a conjecture about the number of 1-dimensional subspaces of  $GF(p)^3$ . Prove your conjecture.

19.5 **By hand** find all the 2-dimensional subspaces of  $GF(3)^3$ .

19.6 Use **GAP** to check your answer to Exercise 19.5.

19.7 Use **GAP** to find the number of 2-dimensional subspaces of  $GF(p)^3$  for  $p = 5, 7$  and  $11$ .

19.8 Make a conjecture about the number of 2-dimensional subspaces of  $GF(p)^3$ .