17 Chapter: Factorization of Polynomials

In this chapter we will investigate the factorization of $x^n - 1$ into its irreducibles over the rational numbers. Recall the GAP commands for creating a polynomial and for factoring this polynomial from the previous chapter. For example:

```
gap> R:= Rationals;
Rationals
gap> x:= X(R,"x");
x
gap> Factors(x^2-1);
[ x-1, x+1 ]
gap> Factors(x^4-1);
[ x-1, x+1, x^2+1 ]
```

Exercises

17.1 Factor $x^n - 1$ into its irreducibles over the rational numbers for $n = 6, 8, 12, 20$ and 30. On the basis of these data make a conjecture about the coefficients of the irreducible factors of $x^n - 1$. Test your conjecture for $n = 40, 50$ and 105. [Gallian, Chapter 17, Computer Exercise 2]

17.2 Notice that your conclusion in Exercise 16.6 of this manual is the Mod p Irreduciblity Test [Gallian, Theorem 17.3]: *Let p be a prime and suppose that $f(x) \in \mathbf{Z}[x]$ with $deg f(x) \geq 1$. Let $\bar{f}(x)$ be the polynomial in $\mathbf{Z}_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x)$ modulo p. If $\bar{f}(x)$ is irreducible over $\mathbf{Z}_p$ and $deg \bar{f}(x) = deg f(x)$, then $f(x)$ is irreducible over $\mathbf{Q}$.* Use this theorem and the `IsIrreducible` command to determine if the following polynomials are irreducible over $\mathbf{Q}$:
a) $x^5 + 9x^4 + 12x^2$
b) $x^4 + x + 1$
c) $x^4 + 3x^2 + 3$
d) $x^5 + 5x^2 + 1$
e) $21x^3 - 3x^2 + 2x + 9$ [Gallian, Chapter 17, Computer Exercise 1]

