14 Chapter: Ideals and Factor Rings

The command in GAP that creates an ideal in the ring R is `Ideal(R,[`*list of generators*`])`. For example to create the ideal I in the ring $\mathbf{Z}_{20}$ generated by 3 and 5 type the following:

```
gap> R:= Integers mod 20;
(Integers mod 20)
gap> e:= Elements(R);
[ ZmodnZObj( 0, 20 ), ZmodnZObj( 1, 20 ), ZmodnZObj( 2, 20 ), ZmodnZObj( 3, 20 ),
  ZmodnZObj( 4, 20 ), ZmodnZObj( 5, 20 ), ZmodnZObj( 6, 20 ), ZmodnZObj( 7, 20 ),
  ZmodnZObj( 8, 20 ), ZmodnZObj( 9, 20 ), ZmodnZObj( 10, 20 ), ZmodnZObj( 11, 20 ),
  ZmodnZObj( 12, 20 ), ZmodnZObj( 13, 20 ), ZmodnZObj( 14, 20 ), ZmodnZObj( 15, 20 ),
  ZmodnZObj( 16, 20 ), ZmodnZObj( 17, 20 ), ZmodnZObj( 18, 20 ), ZmodnZObj( 19, 20 )
gap> I:= Ideal(R, [ e[4], e[6]]);
<two-sided ideal in (Integers mod 20), (2 generators)>
gap> Elements(I);
[ ZmodnZObj( 0, 20 ), ZmodnZObj( 1, 20 ), ZmodnZObj( 2, 20 ), ZmodnZObj( 3, 20 ),
  ZmodnZObj( 4, 20 ), ZmodnZObj( 5, 20 ), ZmodnZObj( 6, 20 ), ZmodnZObj( 7, 20 ),
  ZmodnZObj( 8, 20 ), ZmodnZObj( 9, 20 ), ZmodnZObj( 10, 20 ), ZmodnZObj( 11, 20 ),
  ZmodnZObj( 12, 20 ), ZmodnZObj( 13, 20 ), ZmodnZObj( 14, 20 ), ZmodnZObj( 15, 20 ),
  ZmodnZObj( 16, 20 ), ZmodnZObj( 17, 20 ), ZmodnZObj( 18, 20 ), ZmodnZObj( 19, 20 ) ]
```

In the third command line above `e[4]` denotes the element 3 in $\mathbf{Z}_{20}$ since 3 is the 4th element in the list of elements of $\mathbf{Z}_{20}$. Similarly, `e[6]` denotes the element 5 in $\mathbf{Z}_{20}$ since 5 is listed 6th. Note $I = R$.

Let I be an ideal in a commutative ring R. The *nilradical* of I is defined to be the set $N = \{r \in R \mid r^n \in I \text{ for some positive integer } n\}$.

Exercises

14.1 By hand, find all the ideals in $\mathbf{Z}_{24}$. Which ones are prime?

14.2 For two of the ideals in Exercise 14.1, call them I_1 and I_2, find the intersection of all prime ideals that contain I_1 and find the intersection of all prime ideals that contain I_2.

14.3 Write a short program in GAP that will determine the nilradical of an ideal.

14.4 Using your program from Exercise 14.3, find the nilradicals of I_1 and I_2.

14.5 Repeat Exercises 14.1, 14.2 and 14.4 for the ring $\mathbf{Z}_{900}$.

14.6 Based on your answers to 14.2, 14.4 and 14.5, make a conjecture about the nilradical of a ideal.

14.7 Use your program to find the nilradical of $\langle k \rangle$ in $\mathbf{Z}_n$ for $n = 8, 15, 24$ and for those k that divide n.

