

## 14 Chapter: Ideals and Factor Rings

The command in GAP that creates an ideal in the ring  $R$  is `Ideal(R, [list of generators])`. For example to create the ideal  $I$  in the ring  $\mathbf{Z}_{20}$  generated by 3 and 5 type the following:

```
gap> R:= Integers mod 20;
(Integers mod 20)
gap> e:= Elements(R);
[ ZmodnZObj( 0, 20 ), ZmodnZObj( 1, 20 ), ZmodnZObj( 2, 20 ), ZmodnZObj( 3, 20 ),
  ZmodnZObj( 4, 20 ), ZmodnZObj( 5, 20 ), ZmodnZObj( 6, 20 ), ZmodnZObj( 7, 20 ),
  ZmodnZObj( 8, 20 ), ZmodnZObj( 9, 20 ), ZmodnZObj( 10, 20 ), ZmodnZObj( 11, 20 ),
  ZmodnZObj( 12, 20 ), ZmodnZObj( 13, 20 ), ZmodnZObj( 14, 20 ), ZmodnZObj( 15, 20 ),
  ZmodnZObj( 16, 20 ), ZmodnZObj( 17, 20 ), ZmodnZObj( 18, 20 ), ZmodnZObj( 19, 20 ) ]
gap> I:= Ideal(R, [ e[4], e[6] ]);
<two-sided ideal in (Integers mod 20), (2 generators)>
gap> Elements(I);
[ ZmodnZObj( 0, 20 ), ZmodnZObj( 1, 20 ), ZmodnZObj( 2, 20 ), ZmodnZObj( 3, 20 ),
  ZmodnZObj( 4, 20 ), ZmodnZObj( 5, 20 ), ZmodnZObj( 6, 20 ), ZmodnZObj( 7, 20 ),
  ZmodnZObj( 8, 20 ), ZmodnZObj( 9, 20 ), ZmodnZObj( 10, 20 ), ZmodnZObj( 11, 20 ),
  ZmodnZObj( 12, 20 ), ZmodnZObj( 13, 20 ), ZmodnZObj( 14, 20 ), ZmodnZObj( 15, 20 ),
  ZmodnZObj( 16, 20 ), ZmodnZObj( 17, 20 ), ZmodnZObj( 18, 20 ), ZmodnZObj( 19, 20 ) ]
```

In the third command line above `e[4]` denotes the element 3 in  $\mathbf{Z}_{20}$  since 3 is the 4th element in the list of elements of  $\mathbf{Z}_{20}$ . Similarly, `e[6]` denotes the element 5 in  $\mathbf{Z}_{20}$  since 5 is listed 6th. Note  $I = R$ .

Let  $I$  be an ideal in a commutative ring  $R$ . The *nilradical* of  $I$  is defined to be the set  $N = \{r \in R \mid r^n \in I \text{ for some positive integer } n\}$ .

### Exercises

- 14.1 By hand, find all the ideals in  $\mathbf{Z}_{24}$ . Which ones are prime?
- 14.2 For two of the ideals in Exercise 14.1, call them  $I_1$  and  $I_2$ , find the intersection of all prime ideals that contain  $I_1$  and find the intersection of all prime ideals that contain  $I_2$ .
- 14.3 Write a short program in GAP that will determine the nilradical of an ideal.
- 14.4 Using your program from Exercise 14.3, find the nilradicals of  $I_1$  and  $I_2$ .
- 14.5 Repeat Exercises 14.1, 14.2 and 14.4 for the ring  $\mathbf{Z}_{900}$ .
- 14.6 Based on your answers to 14.2, 14.4 and 14.5, make a conjecture about the nilradical of a ideal.
- 14.7 Use your program to find the nilradical of  $\langle k \rangle$  in  $\mathbf{Z}_n$  for  $n = 8, 15, 24$  and for those  $k$  that divide  $n$ .