13 Chapter: Integral Domains

In this chapter we will determine the number of idempotents and the number of nilpotent elements in the rings $\mathbf{Z}_n$. Recall that an *idempotent* in a ring R is an element r such that $r^2 = r$. A *nilpotent* element $r \in R$ is an element such that $r^m = 0$ for some positive integer m.

Fetch the file "nilpotentCount" off the web site. This file contains a function that counts the number of nilpotent elements in a given ring. The appendix at the end of this chapter contains a print out of this file. (Thanks to Alexander Hulpke for providing a revised version of this function.) GAP has a built in function called `Idempotents` that lists the idempotents in a ring. For an example follow along with the following GAP output. (Recall GAP denotes a mod n in $\mathbf{Z}_n$ by `ZmodnZObj(a,n)`.)

```
gap> M:= Integers mod 6;
(Integers mod 6)
gap> Idempotents(M);
[ ZmodnZObj( 0, 6 ), ZmodnZObj( 1, 6 ), ZmodnZObj( 3, 6 ), ZmodnZObj( 4, 6 ) ]
gap> Size(Idempotents(M));
4
gap> N:= Integers mod 9;
(Integers mod 9)
gap> Size(Idempotents(N));
2
```

The above tells us that $\mathbf{Z}_6$ has 4 idempotents and $\mathbf{Z}_9$ has 2 idempotents.

```
gap> Read("nilpotentCount");
gap> nilpotentCount(M);
1
gap> nilpotentCount(N);
3
```

The above tells us that $\mathbf{Z}_6$ has 1 nilpotent element and $\mathbf{Z}_9$ has 3 nilpotents.

Exercises

13.1 Find the number of idempotents in $\mathbf{Z}_n$ for many values of n. Based on your results answer the following:
a) How many idempotents are in $\mathbf{Z}_n$ when n is a prime-power?
b) How many idempotents are in $\mathbf{Z}_n$ when n is equal to the product of two distinct primes?
c) In general, make a conjecture about the number of idempotents in $\mathbf{Z}_n$ as a function of n.
d) In the case where n is of the form pq where p and q are distinct primes can you see a relationship between the two idempotents that are not 0 and 1? [Gallian, Chapter 13, Computer Exercise 1]

13.2 Find the number of nilpotents in $\mathbf{Z}_n$ for many values of n. Based on your results answer the following:

a) How many nilpotents are in $\mathbf{Z}_n$ when n is a prime-power?
b) How many nilpotents are in $\mathbf{Z}_n$ when n is equal to the product of two distinct primes?
c) In general, make a conjecture about the number of nilpotents in $\mathbf{Z}_n$ as a function of n. [Gallian, Chapter 13, Computer Exercise 2]

13.3 Using GAP, find the number of units in $\mathbf{Z}_n$ for many values of n. Make a conjecture about the number of units in $\mathbf{Z}_n$ as a function of n. (The command `Elements(Units(R))` will list all the units in a given ring R.)

Appendix for Chapter 13

```
nilpotentCount:= function(R)
  local n;
  n:= Size(R);
  return Length(Filtered(Elements(R), i -> IsZero(i^n)));
end;
```

