

## 10 Chapter: Group Homomorphisms

The command `GroupHomomorphismByImages(G,H,[list of generators of G],[list of images of these generators])` in GAP will create the specified homomorphism. For example:

```
gap> S3:= SymmetricGroup(3);
Sym([1..3])
gap> f1:= GroupHomomorphismByImages(S3,S3, [(1,2,3),(1,3)], [(1,3,2),(1,2)]);
[(1,2,3), (1,3)] → [(1,3,2), (1,2)]
```

$f_1$  is the homomorphism  $f_1 : S_3 \rightarrow S_3$  that maps  $(1,2,3)$  to  $(1,3,2)$  and  $(1,3)$  to  $(1,2)$ .

```
gap> Image(f1, (2,3));
(2,3)
gap> Image(f1,(1,2));
(1,3)
```

The above tells us that  $f_1(2,3) = (2,3)$  and  $f_1(1,2) = (1,3)$  (as you can easily check).

```
gap> Size(Image(f1));
6
gap> Kernel(f1);
[()]
```

Thus  $f_1$  is an automorphism. (Again this is easy to check by hand.) For another example consider the following. Read through the GAP commands and be sure you understand the output.

```
gap> f2:= GroupHomomorphismByImages(S3,S3, [(1,2,3),(1,3)], [(1),(1,2)]);
[(1,2,3), (1,3)] → [(), (1,2)]
Size(Image(f2));
2
gap> H:=Image(f2);
Group([(), (1,2)])
gap> Image(f2, (2,3));
(1,2)
gap> Kernel(f2);
Group([(1,2,3)])
```

If you define a map that is not a homomorphism, GAP will return `fail`

```
gap> f3:= GroupHomomorphismByImages(S3,S3, [(1,2,3),(1,3)], [(1,3),(1,2)]);
fail
```

$f_3$  maps  $(1,2,3)$  (an element of order 3) to  $(1,3)$  (an element of order 2) so  $f_3$  is not a homomorphism [Gallian, Theorem 10.1, Part 3].

Recall the group  $D_n$  is a subgroup of  $S_n$  which is generated by a rotation of order  $n$  and a reflection. Thus  $(1, 2, 3, \dots, n)$  and  $(1, n)(2, n-1) \dots (\frac{n}{2}, \frac{n}{2} + 1)$  generate  $D_n$  when  $n$  is even and  $(1, 2, 3, \dots, n)$  and  $(1, n)(2, n-1) \dots (\frac{n-1}{2}, \frac{n-1}{2} + 2)$  generate  $D_n$  when  $n$  is odd. So, for example, every element in  $D_6$  can be written as products of powers of  $(1, 2, 3, 4, 5, 6)$  and  $(1, 6)(2, 5)(3, 4)$  and every element in  $D_7$  can be written as products of powers of  $(1, 2, 3, 4, 5, 6, 7)$  and  $(1, 7)(2, 6)(3, 5)$ .

One way to determine if a homomorphism from the finite group  $G$  to itself is an automorphism is to determine if it is onto. Thus, for example the homomorphism `f1` on the previous page is an automorphism because the image of `f1` is all of  $S_3$ . In the following exercises you will need to use the `GroupHomomorphismByImages` command in `GAP` to find homomorphisms from  $D_n$  to  $D_n$ . You will then need to check if they are automorphisms by checking to see if the kernel contains only the identity or by checking that the image is all of  $D_n$ . Since a homomorphism is completely determined by the image of the generators of a group, you only need to specify where you want to map the two generators of  $D_n$ . You may want to use the fact that the order of the homomorphic image of an element must divide the order of the original element [Gallian, Theorem 10.1, Part 3]. This will help you narrow the possibilities, before using `GAP` to test for automorphisms.

The files “`autoDn`” and “`homoDn`” contain functions that will list all the automorphisms and homomorphisms of a given dihedral group into itself. (Thanks to Alexander Hulpke for providing these functions.) Both files are on the web site as well as in the appendix to this chapter. The following `GAP` output is all the automorphisms and then all the homomorphisms of  $D_6$  into itself:

```
gap> Read("autoDn");
gap> Read("homoDn");
gap> d6:= DihedralGroup(IsPermGroup,12);
Group([ (1,2,3,4,5,6), (2,6)(3,5) ])
gap> autoDn(d6);
[ [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,2,3,4,5,6), (2,6)(3,5) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,2,3,4,5,6), (1,2)(3,6)(4,5) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,2,3,4,5,6), (1,3)(4,6) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,2,3,4,5,6), (1,4)(2,3)(5,6) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,2,3,4,5,6), (1,5)(2,4) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,2,3,4,5,6), (1,6)(2,5)(3,4) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (2,6)(3,5) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (1,2)(3,6)(4,5) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (1,3)(4,6) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (1,4)(2,3)(5,6) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (1,5)(2,4) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (1,6)(2,5)(3,4) ] ]
gap> homoDn(d6);
[ [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (), () ], [ (1,2,3,4,5,6), (2,6)(3,5) ] ->
  [ (), (2,6)(3,5) ], [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (), (1,2)(3,6)(4,5) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (), (1,3)(4,6) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (), (1,4)(2,3)(5,6) ],
  [ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (), (1,4)(2,5)(3,6) ],
```

$[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(), (1,5)(2,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(), (1,6)(2,5)(3,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(2,6)(3,5), ()]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(2,6)(3,5), (2,6)(3,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(2,6)(3,5), (1,4)(2,3)(5,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(2,6)(3,5), (1,4)(2,5)(3,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,2)(3,6)(4,5), ()]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,2)(3,6)(4,5), (1,2)(3,6)(4,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,2)(3,6)(4,5), (1,4)(2,5)(3,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,2)(3,6)(4,5), (1,5)(2,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,2,3,4,5,6), (2,6)(3,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,2,3,4,5,6), (1,2)(3,6)(4,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,2,3,4,5,6), (1,3)(4,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,2,3,4,5,6), (1,4)(2,3)(5,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,2,3,4,5,6), (1,5)(2,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,2,3,4,5,6), (1,6)(2,5)(3,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,3)(4,6), ()]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,3)(4,6), (1,3)(4,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,3)(4,6), (1,4)(2,5)(3,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,3)(4,6), (1,6)(2,5)(3,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,3,5)(2,4,6), (2,6)(3,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,3,5)(2,4,6), (1,2)(3,6)(4,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,3,5)(2,4,6), (1,3)(4,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,3,5)(2,4,6), (1,4)(2,3)(5,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,3,5)(2,4,6), (1,5)(2,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,3,5)(2,4,6), (1,6)(2,5)(3,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,3)(5,6), ()]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,3)(5,6), (2,6)(3,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,3)(5,6), (1,4)(2,3)(5,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,3)(5,6), (1,4)(2,5)(3,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,5)(3,6), ()]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,5)(3,6), (2,6)(3,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,5)(3,6), (1,2)(3,6)(4,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,5)(3,6), (1,3)(4,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,5)(3,6), (1,4)(2,3)(5,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,5)(3,6), (1,4)(2,5)(3,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,5)(3,6), (1,5)(2,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,4)(2,5)(3,6), (1,6)(2,5)(3,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,5)(2,4), ()]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,5)(2,4), (1,2)(3,6)(4,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,5)(2,4), (1,4)(2,5)(3,6)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,5)(2,4), (1,5)(2,4)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,5,3)(2,6,4), (2,6)(3,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,5,3)(2,6,4), (1,2)(3,6)(4,5)]$ ,  
 $[(1,2,3,4,5,6), (2,6)(3,5)] \rightarrow [(1,5,3)(2,6,4), (1,3)(4,6)]$ ,

```

[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,5,3)(2,6,4), (1,4)(2,3)(5,6) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,5,3)(2,6,4), (1,5)(2,4) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,5,3)(2,6,4), (1,6)(2,5)(3,4) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (2,6)(3,5) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (1,2)(3,6)(4,5) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (1,3)(4,6) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (1,4)(2,3)(5,6) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (1,5)(2,4) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6,5,4,3,2), (1,6)(2,5)(3,4) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6)(2,5)(3,4), () ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6)(2,5)(3,4), (1,3)(4,6) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6)(2,5)(3,4), (1,4)(2,5)(3,6) ],
[ (1,2,3,4,5,6), (2,6)(3,5) ] -> [ (1,6)(2,5)(3,4), (1,6)(2,5)(3,4) ] ]
gap> Size(autoDn(d6));
12
gap> Size(homoDn(d6));
64

```

As a homomorphism is completely determined by its image on a set of generators of the group, **GAP** only specifies the image of a set of generators of  $D_6$ . (Be patient when using these functions; they take awhile.)

### *Exercises*

10.1 **By hand** find three automorphism of  $D_4$ . Using **GAP** determine the number of automorphisms of  $D_4$ .

10.2 **By hand** find three homomorphisms from  $D_4$  to  $D_4$  that are not automorphisms. Using **GAP** determine the number of homomorphisms from  $D_4$  to  $D_4$ .

10.3 Repeat Exercises 10.1 and 10.2 for  $D_5$ .

10.4 **Using GAP** repeat Exercises 10.1 and 10.2 for  $D_{19}$ ,  $D_{21}$ ,  $D_{45}$  and  $D_{49}$ .

10.5 Make a conjecture about the number of homomorphisms and the number of automorphisms of  $D_n$  when  $n$  is odd.

10.6 **Using GAP** repeat Exercises 10.1 and 10.2 for  $D_{20}$ ,  $D_{24}$ ,  $D_{48}$  and  $D_{50}$ .

10.7 Make a conjecture about the number of homomorphisms and the number of automorphisms of  $D_n$  when  $n$  is even.

### *Appendix*

The file autoDn:

```

autoDn:= function(G)
local a,b,aims,bims,maps,autos,abims;
a:=G.1; b:=G.2;
aims:=Filtered(Elements(G), i -> Order(a) = Order(i));
bims:=Filtered(Elements(G), i -> Order(b) = Order(i));
abims:= Cartesian(aims,bims);
maps:= List(abims, i -> GroupHomomorphismByImages(G,G,[a,b],i));
maps:= Filtered(maps, i -> i <> fail);
autos:= Filtered(maps, IsInjective);
return autos;
end;

```

The file homoDn:

```

homoDn:= function(G)
local a,b,aims,bims,maps,homos,abims;
a:=G.1; b:=G.2;
aims:=Filtered(Elements(G), i -> IsInt(Order(a) / Order(i)));
bims:=Filtered(Elements(G), i -> IsInt(Order(b) / Order(i)));
abims:= Cartesian(aims,bims);
maps:= List(abims, i -> GroupHomomorphismByImages(G,G,[a,b],i));
homos:= Filtered(maps, i -> i <> fail);
return homos;
end;

```