9 Chapter: Normal Subgroups and Factor Groups

GAP can be used to compute right cosets and factor groups.

```
gap> S6:= SymmetricGroup(6);
Sym( [1..6] )
gap> A6:= AlternatingGroup(6);
Alt( [1..6] )
gap>D6:= DihedralGroup(IsPermGroup, 12);
Group([ (1,2,3,4,5,6), (2,6)(3,5) ])
gap> Z6:= Center(D6);
Group([ (1,4)(2,5)(3,6) ])
```

The above assigns the name `S6` to the symmetric group S_6, `A6` to the subgroup of even permutations, `D6` to the dihedral group D_6, and `Z6` to the center of D_6. To form factor groups we need **normal** subgroups. Test which subgroups are normal:

```
gap> IsNormal(S6,A6);
true
gap> IsNormal(S6,D6);
false
gap> IsNormal(D6,Z6);
true
```

Thus A_6 is normal in S_6 and $Z(D_6)$ is normal in D_6.

```
gap> RightCosets(S6,A6);
[RightCoset(AlternatingGroup[ 1..6 ]), ()),
RightCoset(AlternatingGroup[ 1..6 ]), (5,6))]
```

The above output tells us the right cosets of A_6 in S_6 are A_6 and $A_6(5,6)$. (**Remember:** GAP multiplies permutations from left to right. If your textbook multiplies permutations from right to left then a right coset in GAP will be a left coset using the notation of your textbook, when we are dealing with groups of permutations.) Thus the factor group S_6/A_6 has two elements.

```
gap> Size(FactorGroup(S6,A6));
2
```

Now consider the factor group $D_6/Z(D_6)$.

```
gap> RightCosets(D6,Z6);
[ RightCoset(Group([(1,4)(2,5)(3,6)] ), ()),
RightCoset(Group([(1,4)(2,5)(3,6)] ), (2,6)(3,5)),
RightCoset(Group([(1,4)(2,5)(3,6)] ), (1,5,3)(2,6,4)),
RightCoset(Group([(1,4)(2,5)(3,6)] ), (1,5)(2,4)),
```



```
RightCoset(Group([(1,4)(2,5)(3,6)] ), (1,3,5)(2,4,6)),
RightCoset(Group([(1,4)(2,5)(3,6)] ), (1,3)(4,6)) ]
```

Thus there are 6 right cosets: N, $N(2,6)(3,5)$, $N(1,3)(4,6)$, $N(1,3,5)(2,4,6)$, $N(1,5)(2,4)$ and $N(1,5,3)(2,6,4)$ where $N = Z(D_6)$. So the factor group has 6 elements. Which group of order 6 is $D_6/Z(D_6)$? We will now use GAP to help us answer this question.

```
gap> F:= FactorGroup(D6,Z6);
Group([ f1, f2 ])
gap> IsAbelian(F);
false
```

Since $D_6/Z(D_6)$ is non-Abelian, it can not be isomorphic to $\mathbf{Z}_6$.

```
gap> Read("orderFrequency");
gap> orderFrequency(F);
[Order of element, Number of that order]=[ [ 1, 1 ], [ 2, 3 ], [ 3, 2 ] ]
gap> orderFrequency(SymmetricGroup(3));
[Order of element, Number of that order]=[ [ 1, 1 ], [ 2, 3 ], [ 3, 2 ] ]
```

So $D_6/Z(D_6)$ and S_3 are non-Abelian groups of order 6 with the same number of elements of each order. Two groups that are isomorphic must have the same number of elements of each order. (The converse of this statement is false.) But in this case, this is enough to guarantee that $D_6/Z(D_6)$ and S_3 are isomorphic. [See for example Gallian, Theorem 7.2.])

GAP will also tell you which elements are in a particular coset. For example:

```
gap> Elements(RightCoset(Z6, (2,6)(3,5)));
[ (2,6)(3,5), (1,4)(2,3)(5,6)]
```

Thus the right coset $Z(D_6)(2,6)(3,5) = \{(2,6)(3,5), (1,4)(2,3)(5,6)\}$.

Exercises

9.1 Use GAP to find the right cosets of $Z(D_8)$ in D_8.

9.2 By hand, write out the Cayley table of the factor group $D_8/Z(D_8)$.

You can check your work to Exercise 9.2 by using the GAP command `MultiplicationTable`. For example, to find the Cayley table for S_3 type:

```
gap> e:= Elements(SymmetricGroup(3));
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]
gap> PrintArray(MultiplicationTable( e ));
[ [  1,  2,  3,  4,  5,  6 ],
```



```
[ 2, 1, 4, 3, 6, 5 ],
[ 3, 5, 1, 6, 2, 4 ],
[ 4, 6, 2, 5, 1, 3 ],
[ 5, 3, 6, 1, 4, 2 ],
[ 6, 4, 5, 2, 3, 1 ] ]
```

The GAP output of `PrintArray(MultiplicationTable( e ));` is an n by n array (where n is the order of the group) such that the integer in row i column j equal k if and only if the ith element in the list times the jth element equals the kth element.

9.3 The factor group $D_8/Z(D_8)$ is isomorphic to a group we have used often. Use GAP to help you determine which one.

9.4 Repeat Exercise 9.3 for the factor groups $D_{10}/Z(D_{10})$ and $D_{12}/Z(D_{12})$.

9.5 Based on your results in Exercises 9.3 and 9.4, make a conjecture about the factor group $D_n/Z(D_n)$ when n is even and greater than or equal to 8.

