7 Chapter: Cosets and Lagrange's Theorem

Let G be a group of permutations of a set S. For each $s \in S$ the *stabilizer* of s in G is the subgroup of G equal to $\{g \in G \mid g(s) = s\}$. The *orbit* of s under G is the subset of S equal to $\{g(s) \mid g \in G\}$.

The command `Orbit(G,s)` in GAP will give you the orbit of s under G. The command `Stabilizer(G,s)` creates the subgroup of G that is the stabilizer of s. For example:

```
gap> G:=SymmetricGroup(8);
Sym( [ 1 ..  8 ] )
gap> a:= (1,2,3)(4,5,6);
gap> b:= (7,8);
gap> H:=Subgroup(G,[a,b]);
Group([ (1,2,3)(4,5,6), (7,8) ])
gap> Elements(H);
[ (), (7,8), (1,2,3)(4,5,6), (1,2,3)(4,5,6)(7,8), (1,3,2)(4,6,5),
(1,3,2)(4,6,5)(7,8) ]
gap> Orbit(H,1);
[ 1, 2, 3 ]
gap> Orbit(H,7);
[ 7, 8 ]
gap> Stabilizer(H,1);
Group([ (7,8) ])
gap> Stabilizer(H,7);
Group([ (1,2,3)(4,5,6) ])
gap> Elements(Stabilizer(H,7));
[ (), (1,2,3)(4,5,6), (1,3,2)(4,6,5) ]
```

Careful: Notice that the command `Stabilizer(G,s)` returns a statement describing the stabilizer of s in G in terms of the generators of this stabilizer. To see all the elements in this group you need to use the commands `Elements(Stabilizer(G,s))`.

Exercises

7.1 Find the number of elements in Orbit(G, s) for $G = D_{10}$ and $s = 1, 2, 3$ and 4.

7.2 Find the number of elements in Stabilizer(G, s) for $G = D_{10}$ and $s = 1, 2, 3$ and 4.

7.3 Repeat Exercises 7.1 and 7.2 for D_{49} and D_{50}.

7.4 Make a conjecture about the number elements in Stabilizer(G, s) and in Orbit(G, s) for any $s \in \{1, 2, 3, \ldots, n\}$.

7.5 Explain, geometrically, why your conjecture in Exercise 7.4 is true.

7.6 Generalize the conjecture made in Exercise 7.4 to other finite permutation groups. Use GAP to help you formulate this conjecture.

