6 Chapter: Isomorphisms

An *automorphism* is a map $f : G \to G$ from a group G to itself that is operation preserving and both one-to-one and onto. We are going to examine the possible automorphisms of a finite cyclic group. Suppose we have a finite cyclic group G of order n generated by x. So $G = \{e, x, x^2, x^3, \ldots, x^{n-1}\}$. Define the map $f_k : G \to G$ by $f_k(x^i) = x^{ik}$ for $i = 1, 2, 3, \ldots, n-1$. (Thus for example $f_2(x^3) = x^6$.) Then f_k is a homomorphism. (Show this!) The question we are going to consider is when is f_k an automorphism? Since a finite cyclic group is finite and f_k is a map from G back to G, if we can show f_k is onto, it will have to be one-to-one. Consider the specific example where G is the cyclic group of order 8. As in Chapter 4 we will construct this group as all powers of an 8-cycle.

```
gap> G:= CyclicGroup(IsPermGroup, 8);
Group([ (1,2,3,4,5,6,7,8) ])
gap> Elements(G);
[ (), (1,2,3,4,5,6,7,8), (1,3,5,7)(2,4,6,8), (1,4,7,2,5,8,3,6),
(1,5)(2,6)(3,7)(4,8), (1,6,3,8,5,2,7,4), (1,7,5,3)(2,8,6,4), (1,8,7,6,5,4,3,2) ]
gap> a:= G.1;
(1,2,3,4,5,6,7,8)
gap> f:= x -> x^2;
function( x ) ...  end
gap> H:= Subgroup(G,[f(a)]);
Group([ (1,3,5,7)(2,4,6,8) ])
gap> Elements(H);
[ (), (1,3,5,7)(2,4,6,8), (1,5)(2,6)(3,7)(4,8), (1,7,5,3)(2,8,6,4) ]
gap> Size(H);
4
```

The third command above assigns a to the generator of the cyclic group of order 8. The fourth command defines a function f that takes an element x to x^2. H is the image of the map f. Note that H is a proper subgroup of G, so f is not an automorphism.

```
gap> f:= x -> x^3;
function( x ) ...  end
gap> H:= Subgroup(G,[f(a)]);
Group([ (1,4,7,2,5,8,3,6) ])
gap> Size(H);
8
```

In this case f is an automorphism, since $H = G$.

Exercises

6.1 In the cyclic group of order 10, use GAP to determine which f_k for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ are automorphisms. Based on your results, formulate a conjecture that describes when f_k is an automorphism of the cyclic group of order n.

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6.2 Let G and $\bar{G}$ be groups and let $\Phi : G \to \bar{G}$ be an isomorphism. Then for all $a \in G$ the order of a equals the order of $\Phi(a)$ [Gallian, Theorem 6.2, Part 5]. Is there a connection between your conjecture and this fact? Explain.

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