5 Chapter: Permutation Groups

Consider the puzzle in Figure 5.1 at the end of this chapter. Think of the space in the middle (without a number) as empty. You can slide the other numbers along any of the lines into an empty spot but you can not lift or jump over other numbers. One rearrangement we can do is move the disk in position 1 into the middle then move the disk in position 6 to position 1, then the disk in position 5 to position 6, then the disk in position 4 to position 5, then the disk in position 3 to position 4, and then the disk in the middle to position 3. Denote this rearrangement by r. We could also move the disk in position 1 to the middle, then move the disk in position 2 to position 1, then move the disk in position 3 to position 2 and then move the disk in the middle to the position 3. Convince yourself that all possible rearrangements of Figure 5.1 can be obtained from taking a finite combination of these two rearrangements. These rearrangements form a group. Enter this group into GAP.

```
gap> G:= SymmetricGroup(6);
Sym( [1 ..  6] )
gap> r:=(1,3,4,5,6);
(1,3,4,5,6)
gap> s:=(1,3,2);
(1,3,2)
gap> K:= Subgroup(G, [r,s]);
Group([(1,3,4,5,6), (1,3,2)]);
gap> Elements(K);
```

The elements in the subgroup K describe all the possible arrangements of this puzzle. If you want to know how to get the original arrangement of the puzzle into a particular arrangement, use the `Factorization` command in GAP. For example, to see how to change the original arrangement to the arrangement in Figure 5.2 type

```
gap> Factorization(K,(2,3,4));
x1^-1*x2*x1
```

The `x1` denotes the first generator of K and `x2` denotes the second generator of K. Thus the above output from GAP tells us $(2,3,4) = (1,3,4,5,6)^{-1} * (1,3,2) * (1,3,4,5,6)$. (Remember that GAP multiplies permutations from left to right!) The element $(1,3,4,5,6)^{-1}$ denotes the inverse of $(1,3,4,5,6)$. In terms of the puzzle, $(1,3,4,5,6)^{-1}$ means reversing the loop $(1,3,4,5,6)$, which is the rearrangement $(1,6,5,4,3)$. Note that $(1,3,4,5,6)^{-1} = (1,3,4,5,6)^4$ and $(1,3,2)^{-1} = (1,3,2)^2$.

Exercises

5.1 Indicate what arrangement of the puzzle in Figure 5.1 corresponds to the following permutations. Use GAP to determine how (if possible) to get the following arrangements from the arrangement in Figure 5.1.
a) (4,5,6)
b) (2,3)
c) (1,2)(3,4)

d) (1,2)(3,4)(5,6)
Check the answers to parts a and c by hand. The function `Factorization(K,a)` in GAP will return `fail` if a cannot be expressed in terms of the generators of K. [Gallian, Chapter 5, Computer Exercise 2]

5.2 Repeat Exercise 5.1 for the puzzle in Figure 5.3. Hint: there are two permutations that generate all possible permutations of this puzzle. Check the answer to part d by hand. [Gallian, Chapter 2, Computer Exercise 3]

5.3 Let G be S_{12}. The cycle structure of a permutation is the number of 2-cycles, 3-cycles, etc. it contains when it is written as the product of disjoint cycles. For example (1,2,3)(4,5) and (1,3,6)(2,7) have the same cycle structure. Let $a = (1, 5, 10)$ and $b = (1, 3, 5, 7, 9, 11)$. (Note a and b are elements of G.)
a) What do you think will be the cycle structure of b^2, b^3 and b^6? Check your answer using GAP.
b) Compute ab. What do you think will be the cycle structure of $(ab)^3$ and $(ab)^4$?

Make a cube out of paper or cardboard. Label the 8 vertices of the cube 1-8 as in Figure 5.4. What do you think is the order of the group of rotations of the cube? Call this group G. It is a subgroup of S_8 because each rotation can be described by noting the movement of the 8 vertices. Notice the element $a = (1, 2, 3, 4)(5, 6, 7, 8)$ is in G since it represents a 90 degree rotation about the axis passing through the centers of the front and back faces. Thus a^k is in G for every power k.

```
gap> S:=SymmetricGroup(8);
Sym([1 ..  8])
gap> a:=(1,2,3,4)(5,6,7,8);
(1,2,3,4)(5,6,7,8)
gap> H:= Subgroup(S,[a]);
Group([ (1,2,3,4)(5,6,7,8) ])
gap> Elements(H);
[ (), (1,2,3,4)(5,6,7,8), (1,3)(2,4)(5,7)(6,8), (1,4,3,2)(5,8,7,6) ]
```

Note that $H \neq G$ since, for example, $b = (1, 5, 8, 4)(2, 6, 7, 3)$ is in G.

```
gap> b:=(1,5,8,4)(2,6,7,3);
(1,5,8,4)(2,6,7,3)
gap> M:=Subgroup(S,[a,b]);
gap> Elements(M);
[ (), (2,4,5)(3,8,6), (2,5,4)(3,6,8), (1,2)(3,5)(4,6)(7,8),
(1,2,3,4)(5,6,7,8), (1,2,6,5)(3,7,8,4), (1,3,6)(4,7,5),
(1,3)(2,4)(5,7)(6,8), (1,3,8)(2,7,5), (1,4,3,2)(5,8,7,6),
(1,4,8,5)(2,3,7,6), (1,4)(2,8)(3,5)(6,7), (1,5,6,2)(3,4,8,7),
(1,5,8,4)(2,6,7,3), (1,5)(2,8)(3,7)(4,6), (1,6,3)(4,5,7),
(1,6)(2,5)(3,8)(4,7), (1,6,8)(2,7,4), (1,7)(2,3)(4,6)(5,8),
(1,7)(2,6)(3,5)(4,8), (1,7)(2,8)(3,4)(5,6), (1,8,6)(2,4,7),
(1,8,3)(2,5,7), (1,8)(2,7)(3,6)(4,5) ]
```



```
gap> Size(M);
24
```

Convince yourself that every rotation of the cube is in M so $G = M$. Thus the rotational symmetries of a cube is a subgroup of S_8 of order 24.

Exercises

5.4 Use GAP to describe the symmetry $(1, 8, 3)(2, 5, 7)$ of the cube in terms of the generators a and b.

The function `CycleStructurePerm(a)` in GAP gives the cycle structure of the permutation a. The cycle structure of (1,2,3)(4,5)(6,7) for example, is denoted by [2,1]. (The first spot in the bracket notation denotes the number of 2-cycles, the second spot the number of 3-cycles, etc.) The cycle structure of (1,2,3)(4,5,6,7,8) is denoted by [,1,,1]. This means there are no 2-cycles, one 3-cycle, no 4-cycles and one 5-cycle. The absence of a number after a comma indicates there are no cycles of that length.

```
gap> CycleStructurePerm((1,2,3)(4,5)(6,7));
[ 2, 1 ]
gap> CycleStructurePerm((1,2,3)(4,5,6,7,8));
[ , 1, , 1 ]
```

The following function lists all the elements in a permutation group G that have the cycle structure s:

```
gap> cstruc:= function(G,s)
> return Filtered(Elements(G), x -> CycleStructurePerm(x) = s);
> end;
function( G, s ) ... end
```

(Thanks to Alexander Hulpke for providing this function.) Type this function into GAP or fetch it from the web site. We can now use this function to, for example, find all the elements in S_6 that have one 2-cycle and one 4-cycle:

```
gap> cstruc(SymmetricGroup(6),[1,,1]);
[ (1,2)(3,4,5,6), (1,2)(3,4,6,5), (1,2)(3,5,6,4), (1,2)(3,5,4,6),
(1,2)(3,6,5,4), (1,2)(3,6,4,5), (1,2,3,4)(5,6), (1,2,3,5)(4,6),
(1,2,3,6)(4,5), (1,2,4,3)(5,6), (1,2,4,6)(3,5), (1,2,4,5)(3,6),
(1,2,5,3)(4,6), (1,2,5,6)(3,4), (1,2,5,4)(3,6), (1,2,6,3)(4,5),
(1,2,6,5)(3,4), (1,2,6,4)(3,5), (1,3,4,2)(5,6), (1,3,5,2)(4,6),
(1,3,6,2)(4,5), (1,3)(2,4,5,6), (1,3)(2,4,6,5), (1,3,2,4)(5,6),
(1,3,5,6)(2,4), (1,3,6,5)(2,4), (1,3)(2,5,6,4), (1,3)(2,5,4,6),
(1,3,2,5)(4,6), (1,3,4,6)(2,5), (1,3,6,4)(2,5), (1,3)(2,6,5,4),
(1,3)(2,6,4,5), (1,3,2,6)(4,5), (1,3,4,5)(2,6), (1,3,5,4)(2,6),
(1,4,3,2)(5,6), (1,4,6,2)(3,5), (1,4,5,2)(3,6), (1,4,2,3)(5,6),
(1,4,5,6)(2,3), (1,4,6,5)(2,3), (1,4)(2,3,5,6), (1,4)(2,3,6,5),
```



```
(1,4,6,3)(2,5), (1,4)(2,5,6,3), (1,4)(2,5,3,6), (1,4,2,5)(3,6),
(1,4,3,6)(2,5), (1,4,5,3)(2,6), (1,4)(2,6,5,3), (1,4)(2,6,3,5),
(1,4,2,6)(3,5), (1,4,3,5)(2,6), (1,5,3,2)(4,6), (1,5,6,2)(3,4),
(1,5,4,2)(3,6), (1,5,2,3)(4,6), (1,5,6,4)(2,3), (1,5,4,6)(2,3),
(1,5)(2,3,4,6), (1,5)(2,3,6,4), (1,5,6,3)(2,4), (1,5)(2,4,6,3),
(1,5,2,4)(3,6), (1,5,3,6)(2,4), (1,5)(2,4,3,6), (1,5,4,3)(2,6),
(1,5)(2,6,4,3), (1,5,3,4)(2,6), (1,5)(2,6,3,4), (1,5,2,6)(3,4),
(1,6,3,2)(4,5), (1,6,5,2)(3,4), (1,6,4,2)(3,5), (1,6,2,3)(4,5),
(1,6,5,4)(2,3), (1,6,4,5)(2,3), (1,6)(2,3,4,5), (1,6)(2,3,5,4),
(1,6,5,3)(2,4), (1,6)(2,4,5,3), (1,6,2,4)(3,5), (1,6,3,5)(2,4),
(1,6)(2,4,3,5), (1,6,4,3)(2,5), (1,6)(2,5,4,3), (1,6,3,4)(2,5),
(1,6)(2,5,3,4), (1,6,2,5)(3,4) ]
```

Exercises

5.5 Use GAP to find the number of permutations in S_9 of the following forms:
a) A product of a 4-cycle, and two 2-cycles (for example $(1,2,3,4)(5,6)(7,8)$)
b) A product of a 5-cycle and a 4-cycle
c) A product of three 3-cycles
d) A product of four 2-cycles.

5.6 Recall the command `Centralizer(G,a)` finds the centralizer of an element a in a group G. Find the size of centralizer of each of the following elements in S_9:
a) $(1,2,3,4)(5,6)(7,8)$ and $(5,1,3,4)(2,6)(7,8)$
b) $(1,2,3,4,5)(6,7,8,9)$ and $(1,4,9,6,7)(2,3,5,8)$
c) $(1,2,3)(4,5,6)(7,8,9)$ and $(1,5,8)(2,4,9)(3,6,7)$
d) $(1,2)(3,4)(5,6)(7,8)$ and $(1,9)(2,8)(3,7)(4,6)$.
Based on these answers, for any element $a \in S_n$, make a conjecture about the number of elements in the centralizer of a and the number of element in the centralizer of any permutation in S_n with the same cycle structure as a. Test your conjecture out for some elements of S_7.

5.7 Find a relationship between the answers you obtained in each part of Exercises 5.5 and 5.6 and the order of S_9.

5.8 Pick an element in S_9 and call it a. Compare its cycle structure to the cycle structure of the permutation bab^{-1} for
a) $b = (1,2,3,4,5,6,7,8,9)$
b) $b = (1,2)(3,4)(5,6)(7,8)$
c) $b = (1,2,3,4)(5,6,7,8)$.

5.9 Repeat Exercise 5.8 for a different element a in S_9.

5.10 Make a conjecture about, given two elements a and b in a group of permutations G, how the cycle structure of a and bab^{-1} are related. Test your conjecture for a pair of elements in the dihedral group D_{50}.

5.11 Based on your conjecture in Exercise 5.10, make a conjecture about a relationship between the order of an element a and the order of bab^{-1}.

5.12 Recall the command for finding the order of an element a in GAP is `Order(a)`. Let $a = (1, 2)$. For the elements b in Exercise 5.8 compute the orders of ab and ba. In these three cases is it true that $|ab| = |ba|$?

The elements $r = (1, 3, 4, 5, 6)$ and $s = (1, 3, 2)$ above generated A_6:

```
gap> Size(Group([(1,3,4,5,6), (1,3,2)]));
360
gap> Factorial(6)/2;
360
```

Using the fact that A_n is the only subgroup of S_n of order $|S_n|/2$, we know that r and s generate A_6.

In the following exercises we investigate subgroups of S_n generated by two elements.

Exercises

5.13 Use GAP to help you conjecture what subgroup of S_n is generated by $a = (1, 2)$ and $b = (1, 2, \ldots, n)$.

5.14 For a fixed n, calculate the order of the subgroup of S_n generated by $(1, x)$ and $(1, 2, 3, \ldots, n)$ for various choices of x and n. What conditions on x and n are both necessary and sufficient for $(1, x)$ and $(1, 2, 3, \ldots, n)$ to generate S_n? (Thanks to Daniel Heath at Pacific Lutherian University for providing this exercise.) [Gallian, Chapter 5, Computer Exercise 1]

5.15 Using GAP for at least eight values of n determine the subgroup of S_n generated by the $(n-1)$-cycle $(1, 3, 4, 5, \ldots, n)$ and the 3-cycle $(1, 3, 2)$.

5.16 Using GAP for at least eight values of n determine the subgroup of S_n generated by the $(n-1)$-cycle $(1, 3, 4, 5, \ldots, n)$ and the 4-cycle $(1, 4, 3, 2)$.

5.17 Explain **why** you get different subgroups of S_n in Exercises 5.15 and 5.16 depending on whether n is even or odd and on whether the second generator is a 3 or 4-cycle.

5.18 For a fixed n, calculate the order of the subgroup of S_n generated by $(1, x, y)$ and $(1, 2, 3, \ldots, n)$ for various choices of x and y. Conjecture a necessary and sufficient condition involving x, y, and n so that $(1, x, y)$ and $(1, 2, 3, \ldots, n)$ generate S_n. (Thanks to Daniel Heath at Pacific Lutherian University for providing this exercise.)

At this point, you may find it interesting to note how GAP updates the information about a group (or other constructed object) as it determines characteristics of the group. For example, create the subgroup of S_5 generated by the permutations (1,3) and (1,4,5):

```
gap> g:= Group([(1,3), (1,4,5)]);
Group([ (1,3), (1,4,5) ])
```

The command `KnownAttributesOfObject` returns the current information GAP contains on your group:

```
gap> KnownAttributesOfObject(g);
[ "LargestMovedPoint", "GeneratorsOfMagmaWithInverses",
"MultiplicativeNeutralElement" ]
```

(You can use the the command line help, ? followed by the command, to see what any of these three characteristics mean.) If we now compute with `g` and then reuse the command `KnownAttributesOfObject`, we see that GAP now has more information about our group `g`:

```
gap> Size(g);
24
gap> KnownAttributesOfObject(g);
[ "Size", "OneImmutable", "LargestMovedPoint", "NrMovedPoints", "MovedPoints",
"GeneratorsOfMagmaWithInverses", "MultiplicativeNeutralElement", "Pcgs",
"GeneralizedPcgs", "StabChainMutable", "StabChainOptions" ]
```

In determining the order of our group, GAP went through constructions that determined these 8 new attributes.

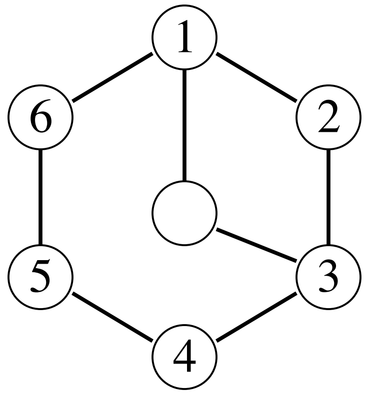

Figure 5.1

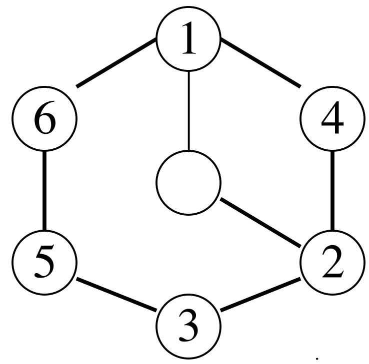

Figure 5.2

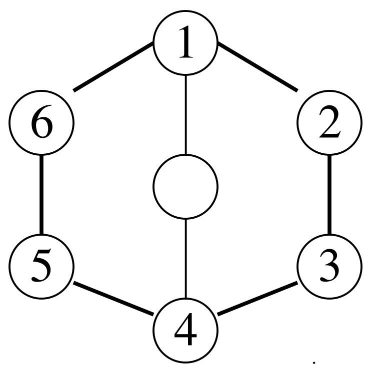

Figure 5.3

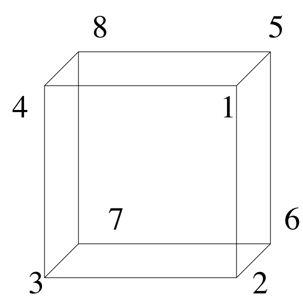

Figure 5.4