4 Chapter: Cyclic Groups

We can describe a cyclic group of order n, as the group of all powers of the n-cycle $(1, 2, \ldots, n)$. The following sets up the cyclic group of order 6 as the group of all powers of a 6-cycle.

```
gap> c6:= CyclicGroup(IsPermGroup, 6);
Group([ (1,2,3,4,5,6) ])
gap> Elements(c6);
[ (), (1,2,3,4,5,6), (1,3,5)(2,4,6), (1,4)(2,5)(3,6), (1,5,3)(2,6,4), (1,6,5,4,3,2) ]
gap> a:= c6.1;
(1,2,3,4,5,6)
gap> Elements(Subgroup(c6,[a^2]));
[ (), (1,3,5)(2,4,6), (1,5,3)(2,6,4) ]
```

The third command above assigns the name a to the generator of the group c6. The fourth command is asking GAP for the elements in the subgroup of c6 generated by the element a^2 (that is, the subgroup consisting of all powers of a^2). The output from this command says that this subgroup contains only the identity, a^2 and a^4. As was done in Chapter 3, we can also use GAP to determine a subgroup generated by two or more elements. For example the following shows the subgroup of c6 generated by both a^2 and a^3:

```
gap> Elements(Subgroup(c6,[a^2, a^3]));
[ (), (1,2,3,4,5,6), (1,3,5)(2,4,6), (1,4)(2,5)(3,6), (1,5,3)(2,6,4), (1,6,5,4,3,2) ]
```

This is of course all of c6.

Exercises

4.1 Use GAP to list the subgroups of the following groups. *Hint:* The above explanation shows how to generate subgroups of a group. Find which subgroups are generated by one element, which by two elements, etc.
a) D_4
b) the cyclic subgroup of D_8 generated by $(1, 2, 3, 4, 5, 6, 7, 8)$.
Draw the subgroup lattice for each of the above groups.

4.2 Let G be the cyclic group generated by an element a of order n. By the Fundamental Theorem of Cyclic Groups there is exactly one subgroup of G of order k for each k that divides n. In addition, by the Fundamental Theorem of Cyclic Groups, every subgroup of a cyclic group is cyclic. So this subgroup of order k must be cyclic. Use GAP to find the smallest subgroup of G containing
a. a^4 and a^6 when $n = 30$
b. a^{10} and a^2 when $n = 30$
c. a^{15} and a^2 when $n = 30$
d. a^9 and a^{12} when $n = 30$
e. a^8 and a^{12} when $n = 30$
In each part a-e find an integer t such that this smallest subgroup is $\langle a^t \rangle$.

4.3 Repeat Exercise 4.2 for $n = 60$.

4.4 Formulate a conjecture that describes the smallest subgroup of a cyclic group G of order n that contains a^i and a^j for any positive integers i, j and n, where a is a generator of G and i and j are less than n. (You may have to do many more examples before you arrive at a conjecture.)

4.5 Again let G be the cyclic group generated by an element a of order n. Use GAP to find the smallest positive integer t such that $\langle a^t \rangle$ is the subgroup:
a. $\langle a^4 \rangle \cap \langle a^6 \rangle$ when $n = 30$
b. $\langle a^{10} \rangle \cap \langle a^2 \rangle$ when $n = 30$
c. $\langle a^{15} \rangle \cap \langle a^2 \rangle$ when $n = 30$
Hint: Type `?Intersection` at the GAP prompt to see how to find intersections in GAP.

4.6 Repeat Exercise 4.5 for $n = 60$.

4.7 Formulate a conjecture that describes the smallest subgroup of a cyclic group G of order n that contains $\langle a^i \rangle \cap \langle a^j \rangle$ for any integers i, j and n, where a is a generator of G and i and j are less than n. (You may have to do many more examples before you arrive at a conjecture.)

In the remainder of this chapter you will need the file "orderFrequency". Fetch this file off the web site. This file contains the function `orderFrequency` which will tell you the number of elements of each order in a given group. For example:

```
gap> Read("orderFrequency");
gap> orderFrequency(c6);
[Order of element, Number of that order]=[ [ 1, 1 ], [ 2, 1 ], [ 3, 2 ],
[ 6, 2 ] ]
```

The output tells us that the cyclic group of order 6 has one element of order 1, one of order 2, two of order 3 and two of order 6.

Exercises

4.8 Find the number of elements of each order in the cyclic groups of order 75 and 90.

4.9 Find the number of elements of each order in the dihedral groups D_{17}, D_{25}, D_{33} and D_{49}. Make a conjecture about the number of elements of order 2 in D_n.

4.10 Find the number of elements of order 2 in the dihedral groups D_{18} D_{26}, D_{34} and D_{50}. Make a conjecture about the number of elements of order 2 in D_n. (Be careful that your conjectures for Exercises 4.9 and 4.10 are not contradictory.)

4.11 Do you see any relationship between the orders of elements in a group and the order of the group?

Let C_m denote the cyclic group of order m generated by an m-cycle. For any pair of positive integers m and n, let $C_m \oplus C_n = \{(a,b) \mid a \in C_m, b \in C_n\}$. For any pair of elements (a,b) and (c,d) in $C_m \oplus C_n$, define $(a,b)*(c,d) = (a*c, b*d)$. This binary operation makes $C_m \oplus C_n$ into a group. We can set up groups of this form in GAP. For example the following creates the group $G = C_4 \oplus C_6$:

```
gap> c4:= CyclicGroup(IsPermGroup, 4);
Group([ (1,2,3,4) ])
gap> c6:= CyclicGroup(IsPermGroup, 6);
Group([ (1,2,3,4,5,6) ])
gap> G:= DirectProduct(c4, c6);;
gap> IsCyclic(G);
false
```

Note that even though C_4 and C_6 are cyclic groups, the group $C_4 \oplus C_6$ is not cyclic.

Exercises

4.12 Determine whether or not $C_m \oplus C_n$ is cyclic for $(m,n) = (2,2), (2,3), (2,4), (2,5), (3,4), (3,5), (3,6), (3,7), (3,8), (3,9)$, and $(4,8)$. On the basis of this output, guess how m and n must be related for $C_m \oplus C_n$ to be cyclic. [Gallian, Chapter 4, Computer Exercise 2]

Appendix for Chapter 4

The following is the file "orderFrequency" which is used in this chapter. (Thanks to Alexander Hulpke for providing a revised version.)

```
orderFrequency:= function(g)
local h,w;
w:= [];
w:= h -> Collected(List(Elements(h), Order));
Print("[Order of element, Number of that order]=");
return w(g);
end;
```