St. Louis University
Fall 2007

Putnam Mathematical Competition

Math 293–01
Prof. G. Marks

The Putnam Exam is the premier national undergraduate mathematics competition, which will next be held on Saturday, December 1, 2007. (The registration deadline is two months earlier.) Approximately 3,500 undergraduate students representing 500 colleges and universities throughout the U.S. and Canada are expected to compete. Registration is free. In addition to awarding cash prizes (up to $2,500) to top scorers, the Mathematical Association of America will publish the names and schools of the top 500 students.

In 2006, a score of 15/120 would have put you in the top 500 nationwide. The exam, needless to say, is not easy! *If you can solve even one problem completely correctly, you will be making a significant contribution to the SLU team.* A good performance can bring great prestige to both you and the university.

Interested students should contact the local supervisor, Greg Marks (**marks@slu.edu**), and might consider enrolling in Prof. Marks's one-unit P/NP course Math 293, on mathematical problem-solving. Please take notice of the rules for the competition, printed below.

> History. The competition began in 1938 and is designed to stimulate a healthful rivalry in mathematical studies in the colleges and universities of the United States and Canada. It exists because Mr. William Lowell Putnam had a profound conviction in the value of organized team competition in regular college studies. Mr. Putnam, a member of the Harvard class of 1882, wrote an article for the December 1921 issue of the Harvard Graduates' Magazine in which he described the merits of an intellectual intercollegiate competition. To establish such a competition, his widow, Elizabeth Lowell Putnam, in 1927 created a trust fund known as the William Lowell Putnam Intercollegiate Memorial Fund. The first competition supported by this fund was in the field of English and a few years later a second experimental competition was held, this time in mathematics between two institutions. It was not until after Mrs. Putnam's death in 1935 that the examination assumed its present form and was placed under the administration of the Mathematical Association of America.
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> Description. The examination will be constructed to test originality as well as technical competence. It is expected that the contestant will be familiar with the formal theories embodied in undergraduate mathematics. It is assumed that such training, designed for mathematics and physical science majors, will include somewhat more sophisticated mathematical concepts than is the case in minimal courses. Thus the differential equations course is presumed to include some references to qualitative existence theorems and subtleties beyond the routine solution devices. Questions will be included that cut across the bounds of various disciplines, and self-contained questions that do not fit into any of the usual categories may be included. It will be assumed that the contestant has acquired a familiarity with the body of mathematical lore commonly discussed in mathematics clubs or in courses with such titles as "survey of the foundations of mathematics." It is also expected that the self-contained questions involving elementary concepts from group theory, set theory, graph theory, lattice theory, number theory, and cardinal arithmetic will not be entirely foreign to the contestant's experience.
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> William Lowell Putnam Prizes. Prizes will be awarded to the departments of mathematics of the institutions with the five winning teams. In addition, there will be prizes awarded to each of the members of these teams. The five highest ranking individuals are designated Putnam Fellows by the Mathematical Association of America. Prizes will be awarded to each of these individuals and to each of the next twenty highest ranking contestants.
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> Rules. The competition is open only to regularly enrolled undergraduates, in colleges and universities of the United States and Canada, who have not yet received a college degree. No individual may participate in the competition more than four times. An eligible entrant who is also a high school student must be informed of this four time limit. A college or university with at least three registered entrants obtains a team rank through the positions achieved by three designated individual contestants. No collaboration or outside assistance is permitted during the examination. Each contestant, even if designated as a team member, must work independently on the examination questions.
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> Please note that there are no provisions for "unofficial" entrants.
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> The local supervisor must be a regular faculty member.

Sample Exam
The Sixty-Seventh William Lowell Putnam Mathematical Competition
Saturday, December 2, 2006

Problems A1 through A6 were given during the 3-hour morning session of the exam; problems B1 through B6 were given during the 3-hour afternoon session.

A1. Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leqslant 36(x^2 + y^2).$$

A2. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

A3. Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geqslant 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

A4. Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

(i) $\pi(k) > \pi(k+1)$ for $k = 1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;

(iii) $\pi(k-1) < \pi(k)$ for $k = n$.

(For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

A5. Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \ldots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer, and determine its value.

A6. Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

B1. Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

B2. Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leqslant \frac{1}{n+1}.$$

B3. Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

B4. Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leqslant k \leqslant n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$.)

B5. For each continuous function $f\colon [0,1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(x) = \int_0^1 x\,(f(x))^2\,dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

B6. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$