# SOLUTIONS - HWK 9

Ch 3.4

2) Let $\arg_0(z)$ be the branch of arg with values in $(0, 2\pi]$

Then $\arg_0(z - i - 1)$ has values $0$ and $\frac{3\pi}{2}$ on the horizontal and vertical boundary of the region.
So $-\frac{20}{3\pi}\arg_0(z-i-1)$ has values $0$ and $-10$, so

$$\phi(z) = -\frac{20}{3\pi}\arg_0(z-i-1) + 10$$ has the desired boundary values.

At $z = 0$, $\arg_0(-i-1) = \frac{5\pi}{4}$, so $\phi(0) = -\frac{20}{3\pi}\left(\frac{5\pi}{4}\right) + 10$

$$\phi(0) = -\frac{25}{3} + 10 = \frac{5}{3}$$

(Note $\frac{5}{3} = \frac{1}{6}\cdot 10$, and $z=0$ is on the ray $\frac{1}{6}^{th}$ of the way from 0 to 10)

4) We'll write it as $A\cdot \mathrm{Arg}(z+1) + B\cdot \mathrm{Arg}(z-2) + C$.
There are three cases when $z$ is real:

$z < -1$ : $A\cdot\pi + B\cdot\pi + C = 0$

$-1 < z < 2$ : $A\cdot 0 + B\cdot\pi + C = \pi$

$2 < z$ : $A\cdot 0 + B\cdot 0 + C = 0$

So $C = 0$, $B = 1$, and $A = -1$ works,

$$\phi(z) = \mathrm{Arg}(z-2) - \mathrm{Arg}(z+1)$$

and $\phi(2,3) = \mathrm{Arg}(3i) - \mathrm{Arg}(3+3i) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

6) The constant function $\phi(z) = 30$ is harmonic and has the correct boundary value.

4.1) 8) Let $\gamma_1 = (1-t)(-2+2i) + t(-1) = -2+2i+t-2it$, $0 \le t \le 1$
and $\gamma_2 = e^{-it}$, $\pi \le t \le 2\pi$
Then $\Gamma = \gamma_1 + \gamma_2$. For a parameterization of $\Gamma$ using a single variable, could do

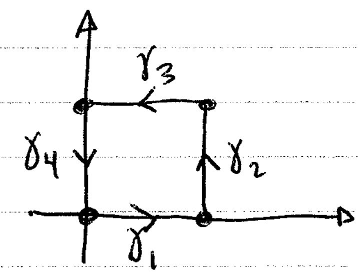
$$\Gamma(t) = \begin{cases} -2+2i+t-2it, & 0 \le t \le 1 \\ e^{-i\pi t}, & 1 \le t \le 2 \end{cases} \quad \text{for } t \in [0,2]$$

Then $-\Gamma$ is parameterized as

$$-\Gamma(t) = \begin{cases} e^{i\pi t} & -2 \le t \le -1 \\ -2+2i-t+2it & -1 \le t \le 0 \end{cases} \quad \text{for } t \in [-2,0]$$

4.2) 8) Parameterize as

$$\begin{array}{lll} \gamma_1 = t & 0 \le t \le 1 & \gamma_1' = 1 \\ \gamma_2 = 1+it & 0 \le t \le 1 & \gamma_2' = i \\ \gamma_3 = 1+i-t & 0 \le t \le 1 & \gamma_3' = -1 \\ \gamma_4 = i-it & 0 \le t \le 1 & \gamma_4' = -i \end{array}$$




$$\int_{\gamma_1} e^z dz = \int_0^1 e^t dt = e^t\Big|_0^1 = e-1$$

$$\int_{\gamma_2} e^z dz = \int_0^1 e^{1+it}\, i\,dt = e^{1+it}\Big|_0^1 = e^{1+i} - e$$

$$\int_{\gamma_3} e^z dz = \int_0^1 e^{1+i-t}(-1)dt = e^{1+i-t}\Big|_0^1 = e^i - e^{1+i}$$

$$\int_{\gamma_4} e^z dz = \int_0^1 e^{i-it}(-i)dt = e^{i-it}\Big|_0^1 = 1 - e^i$$

So $$\int_C e^z dz = e - 1 + e^{1+i} - e + e^i - e^{1+i} + 1 - e^i = 0$$

4.2)
14a) The triangle inequality gives $|z^2 - i| \geq \big||z^2| - |i|\big| = 9 - 1 = 8$ when $|z| = 3$. Then $\left|\frac{1}{z^2-i}\right| \le \frac{1}{8}$, so

$$\left|\int_C \frac{dz}{z^2-i}\right| \le \underbrace{\int_C \left|\frac{dz}{z^2-i}\right|}_{} \le \frac{1}{8} \cdot \text{length}(C) = \frac{3\pi}{4}$$

(This is just a way to remember the Theorem... no real meaning)

Ch 4.2

15) Let $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$.

Then

$$\left|\sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})\right| \le \sum_{i=1}^{n} \left|f(t_i)(t_i - t_{i-1})\right| = \sum_{i=1}^{n} |f(t_i)|(t_i - t_{i-1})$$

The left hand side is a Riemann sum for $\left|\int_a^b f(t)dt\right|$,

and the right hand side is a Riemann sum for $\int_a^b |f(t)|dt$

Since the inequality holds for every Riemann sum, it holds in the limit as the mesh $\to 0$, so $\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt$

$\square$

Another approach, without Riemann sums:

Put $\int_a^b f(t)dt = r_0 e^{i\theta_0}$ for some $r_0, \theta_0$.

Then

$$\left|\int_a^b f(t)dt\right| = \left|e^{-i\theta_0}\int_a^b f(t)dt\right| = \left|\int_a^b e^{-i\theta_0} f(t)dt\right|$$

But $\int_a^b e^{-i\theta_0} f(t)dt = r_0$, so $\left|\int_a^b e^{-i\theta_0} f(t)dt\right| = r_0 = Re\int_a^b e^{-i\theta_0} f(t)dt$

Then

$$\left|\int_a^b f(t)dt\right| = Re\int_a^b e^{-i\theta_0} f(t)dt \overset{\circledast}{=} \int_a^b Re\left(e^{-i\theta_0} f(t)\right)dt$$

$$\le \int_a^b \left|e^{-i\theta_0} f(t)\right|dt = \int_a^b |f(t)|dt$$

Where the $\circledast$ step uses $Re\int_a^b g(t)dt = \int_a^b Re\, g(t)dt$ is easy to prove, using $\int_a^b g(t)dt = \int_a^b Re(g(t))dt + i\int_a^b Im(g(t))dt$.

Ch 4.3

6) Let $C_{\alpha\beta}$ be the contour as shown in the picture, from $\alpha$ to $\beta$. Then

$$\int_{C_{\alpha\beta}} \frac{dz}{z-z_0} = \operatorname{Log}(z-z_0)\Big|_\alpha^\beta = \operatorname{Log}(\beta-z_0) - \operatorname{Log}(\alpha-z_0)$$

$$= \log|\beta-z_0| + i\operatorname{Arg}(\beta-z_0) - \log|\alpha-z_0| - i\operatorname{Arg}(\alpha-z_0)$$

As $\beta \to \tau$, $\log|\beta-z_0| \to \log|\tau-z_0|$
and $\operatorname{Arg}(\beta-z_0) \to \pi$

As $\alpha \to \tau$, $\log|\alpha-z_0| \to \log|\tau-z_0|$
and $\operatorname{Arg}(\alpha-z_0) \to -\pi$

So, letting $\beta \to \tau$, $\alpha \to \tau$ gives

$$\int_{C_{\alpha\beta}} \frac{dz}{z-z_0} \longrightarrow \log|\tau-z_0| + i\pi - \log|\tau-z_0| + i\pi = 2\pi i$$

So $\int_C \frac{dz}{z-z_0} = 2\pi i$

7) Suppose the circle is centered at $a+bi$, and $z_0$ is outside $C$.

If $\operatorname{Re} z_0 \le a$ then $\operatorname{Log}(z-z_0)$ is an antiderivative for $\frac{1}{z-z_0}$ on $C$.

If $\operatorname{Re} z_0 > a$, then $\mathcal{L}_0(z-z_0)$ is an antiderivative for $\frac{1}{z-z_0}$ on $C$, where $\mathcal{L}_0$ is the branch of log with branch cut on the positive real axis.

Since $C$ is a loop, in both cases $\int_C \frac{dz}{z-z_0} = 0$.

12) Let $\gamma$ be the straight line segment from $z_1$ to $z_2$, so $\text{length}(\gamma) = |z_2 - z_1|$
Since $|f'(z)| \le M$ for all $z \in \gamma$,

$$|f(z_2) - f(z_1)| = \left|\int_\gamma f'(z)\,dz\right| \le M \cdot \text{length}(\gamma) = M|z_2 - z_1|$$