10/30/09 Math 451 – Exam 2 Name: KEY

There are 7 questions, worth a total of 100 points.

(20) 1. Contour integrals:

(a) Compute $\int_C \bar{z}\,dz$,
where C is the unit circle $|z| = 1$, traversed once counterclockwise.

Let $z(t) = e^{it}$, $0 \le t \le 2\pi$ be a param. of C.
Then $z'(t) = ie^{it}$ and

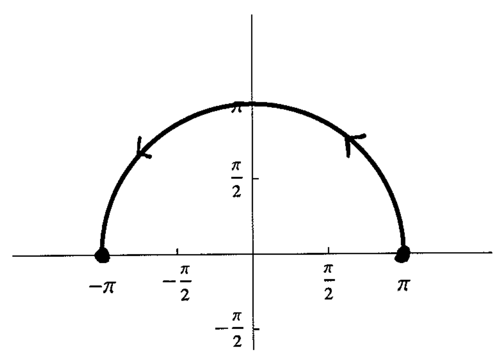
$$\int_C \bar{z}\,dz = \int_0^{2\pi} \overline{e^{it}}\, ie^{it}\,dt = \int_0^{2\pi} e^{-it}\cdot i \cdot e^{it}\,dt = \int_0^{2\pi} i\,dt$$

$$= it\Big|_0^{2\pi} = \boxed{2\pi i}$$

(b) Compute $\int_\gamma e^z\,dz$,
where γ is the upper half of the circle $|z| = \pi$, traversed once counterclockwise as shown.

e^z has antiderivative e^z,

So $$\int_\gamma e^z\,dz = \boxed{e^{-\pi} - e^{\pi}}$$



(10) 2. Find the principal value of $(-1)^{1/\pi}$.

$$(-1)^{1/\pi} = e^{\frac{1}{\pi}\mathrm{Log}(-1)} = e^{\frac{1}{\pi} i \mathrm{Arg}(-1)} = e^{\frac{1}{\pi}\cdot i\pi} = e^{i}$$

$$\left(= \cos 1 + i \sin 1 \right)$$

(15) 3. Find a function $\phi(x, y)$ on the region shown, which has boundary conditions $\phi = 0$ on the ray $y = -x + 1$, and $\phi = 100$ on the ray $y = x - 1$. Sketch a few of its level curves on the picture.

Let $\varphi(x,y) = a \cdot \mathrm{Arg}(z-1) + b$

Then $a \cdot \left(-\frac{\pi}{4}\right) + b = 0$

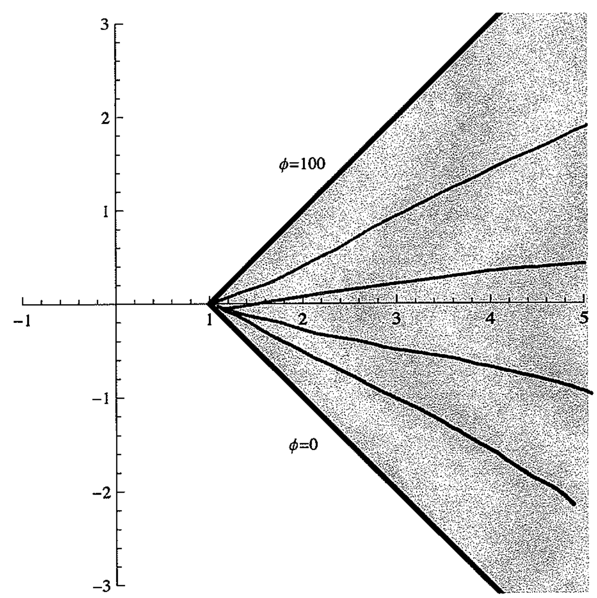
and $a \cdot \left(\frac{\pi}{4}\right) + b = 100$

so $b = a\frac{\pi}{4}$

$a \cdot \frac{\pi}{4} + a \cdot \frac{\pi}{4} = 100$

$a = \frac{200}{\pi} \qquad b = 50$

$\varphi(x,y) = \frac{200}{\pi} \mathrm{Arg}(z-1) + 50$



(15) 4. Find the one point where $f(x+iy) = \overbrace{x^3}^{u}+\overbrace{(1-y)^3}^{v}i$ is differentiable, and find its derivative at that point.

CR equations

$$\frac{\partial u}{\partial x} = 3x^2 \qquad \frac{\partial v}{\partial y} = -3(1-y)^2$$

Notice $3x^2 \geq 0$ and $-3(1-y)^2 \leq 0$

So $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ only when $3x^2 = 0$ and $-3(1-y)^2 = 0$

So $x = 0$, $y = 1$

Only differentiable at $z = i$

$$f'(i) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3x^2 + i \cdot 0 = 0 \quad \text{when } x = 0.$$

(15) 5. Give an example of points $z_1, z_2 \in \mathbb{C}$ so that

$$\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}(z_1) + \mathrm{Log}(z_2).$$

Calculate both sides to show your example works.

Let $z_1 = z_2 = -i$

$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(-1) = i\,\mathrm{Arg}(-1) = i\pi$

$\mathrm{Log}(z_1) = \mathrm{Log}(z_2) = \mathrm{Log}(-i) = i\,\mathrm{Arg}(-i) = i\left(-\frac{\pi}{2}\right)$

So $\mathrm{Log}(z_1) + \mathrm{Log}(z_2) = -i\pi$

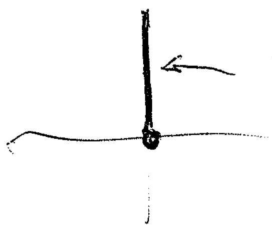
(15) 6. Show that $f(z) = \text{Log}(iz) - \frac{i\pi}{2}$ is a branch of the logarithm and find its branch cuts.

$$e^{f(z)} = e^{\text{Log}(iz) - i\frac{\pi}{2}} = e^{\log iz} \cdot e^{-i\frac{\pi}{2}} = (iz)(-i) = z$$

so $f(z)$ is a branch of logarithm.

f has branch cuts when iz is real and negative,

so $iz = -b$, $z = ib$ for $b \geq 0$



branch cuts on the positive imaginary axis.

(10) 7. Suppose that f is an entire function (analytic on $\mathbb{C}$), and:

- $f'(z) = a$ for all $z \in \mathbb{C}$.
- $f(0) = 0$.

Prove that $f(z) = az$.

Let $g(z) = f(z) - az$

Then g is entire, and $g'(z) = f'(z) - a = 0$ for all z.

So g is constant, $g(z) = C$.

Now $C = g(0) = f(0) - a \cdot 0 = 0$ so $C = 0$.

Then $0 = g(z) = f(z) - az$ so $f(z) = az$

Alternately, we showed if $f' = g'$ then $f = g + C$.

So, ~~also~~ let $g(z) = az$. Then $f'(z) = a = g'(z)$

so $f(z) = az + C$, then $0 = f(0) = a \cdot 0 + C$ so $C = 0$.

So $f(z) = az$