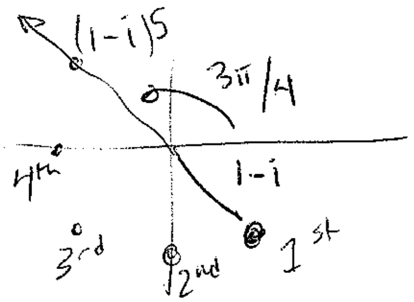
9/23/09 Math 266 – Exam 1 Name: KEY

There are 10 questions, worth a total of 100 points.

(10) 1. Find $\operatorname{Arg}(1-i)^5$

$$\arg(1-i)^5 = 5\arg(1-i) = 5\left(-\frac{\pi}{4}\right) + 2\pi k = -\frac{5\pi}{4} + 2\pi k$$

So $$\operatorname{Arg}(1-i)^5 = -\frac{5\pi}{4} + 2\pi = \frac{3\pi}{4}$$



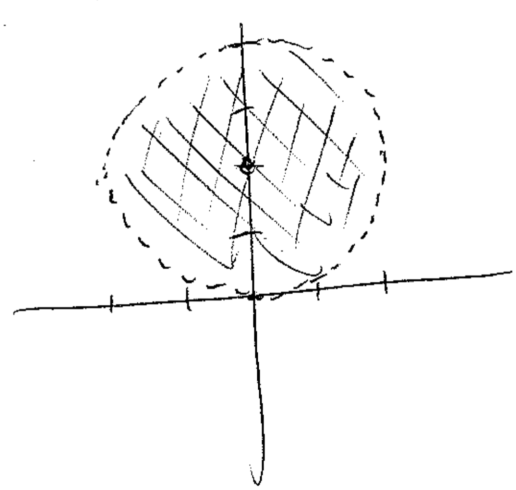
(10) 2. Solve for z:

$iz + 3 = 4z - 2 - i$

$$iz - 4z = -5 - i$$

$$z = \frac{-5-i}{i-4} \cdot \frac{-i-4}{-i-4} = \frac{5i - 1 + 4i + 20}{17} = \frac{9i + 19}{17}$$

(10) 3. Sketch the set $|z - 2i| < 2$.



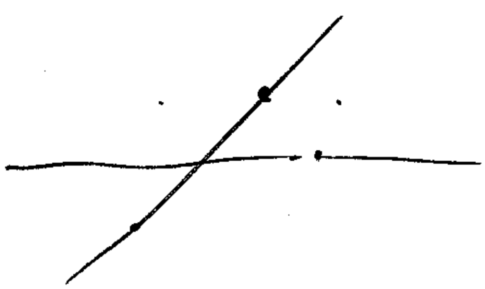
(10) 4. Prove: If $\operatorname{Im} z > 0$ then $\operatorname{Im}(1/z) < 0$.

Suppose $\operatorname{Im} z > 0$, $z = a + bi$ with $b > 0$.

$$\frac{1}{z} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

Since $b > 0$, $\operatorname{Im} \frac{1}{z} = -\frac{b}{a^2+b^2} < 0$.

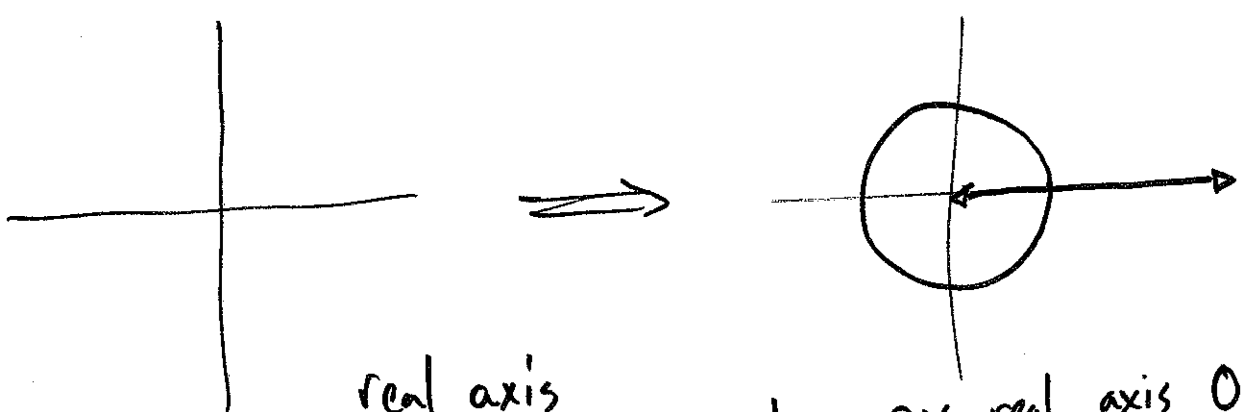
(10) 5. Describe how the map $f(z) = -\bar{z}$ acts on the Riemann sphere.



Refl. in the vertical plane through the Im axis.

(or the x_2-x_3 plane)

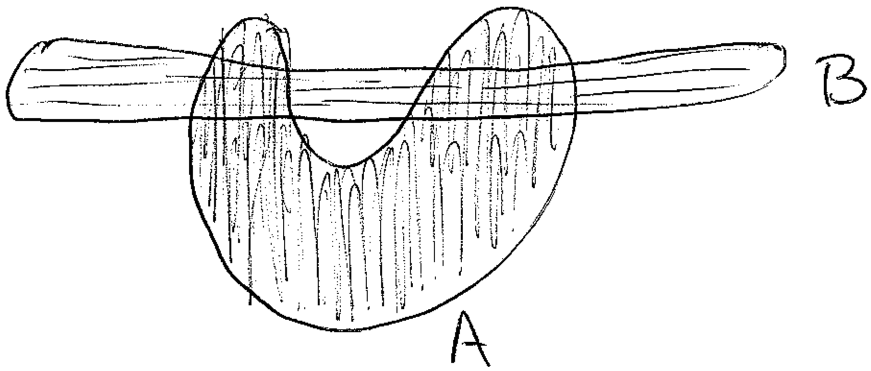
(10) 6. Let $f(z) = e^z$. What does f do to the real and imaginary axes. Explain, then sketch the image of both axes under the mapping f.



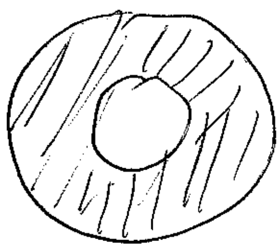
real axis $-\infty$ to ∞ goes to pos. real axis 0 to ∞.

Im axis wraps around the unit circle (∞'ly many times)

(10) 7. (a) Give an example of two connected sets whose intersection is not connected.



(b) Give an example of a connected set whose complement is not connected.



(10) 8. Discuss the convergence of the sequence $z_n = (bi)^n$ as $n \to \infty$, where b is a real number.

When $|b|>1$, $z_n \to \infty$ (-1 for "doesn't converge")

When $|b|=1$, z_n doesn't converge

When $|b|<1$, $z_n \to 0$

(10) 9. Suppose ω is a root of unity.

(a) Prove that $1/\omega = \bar{\omega}$.

(b) Prove that $\bar{\omega}$ is a root of unity.

Write $\omega = e^{ik2\pi/n} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$

Then $\bar{\omega} = \cos\left(\frac{2\pi k}{n}\right) - i\sin\left(\frac{2\pi k}{n}\right)$

$= \cos\left(-\frac{2\pi k}{n}\right) + i\sin\left(-\frac{2\pi k}{n}\right) = e^{-i2\pi k/n}$

$= \frac{1}{\omega}$

Or:

$\omega^n = 1$

so $|\omega^n| = |\omega|^n = 1$

so $|\omega| = 1$

Then $\omega\bar{\omega} = |\omega|^2 = 1$

so $\bar{\omega} = \frac{1}{\omega}$

$\bar{\omega}^n = \overline{\omega^n} = \bar{1} = 1$

(10) 10. Show that $f(z) = |z|^2$ is differentiable at $z = 0$ but nowhere else.

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{(z+h)(\overline{z+h}) - z\bar{z}}{h}$$

$$= \lim_{h\to 0} \frac{h\bar{z} + z\bar{h} + h\bar{h}}{h} = \lim_{h\to 0} \bar{z} + \frac{\bar{h}}{h} z + \bar{h}$$

If $z = 0$, this limit exists and $= 0$.

If $z \neq 0$, let $h = a$, so $\frac{\bar{h}}{h} = 1$, $\lim_{h\to 0} \bar{z} + z + \bar{h} = \bar{z} + z$

let $h = ib$, so $\frac{\bar{h}}{h} = -1$, $\lim_{h\to 0} \bar{z} - z + \bar{h} = \bar{z} - z$

These are not equal, so the limit D.N.E.

$(\cancel{\bar{z}} + z = \cancel{\bar{z}} - z \Rightarrow 2z = 0 \Rightarrow z = 0)$