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Some Results Concerning Surface Groups in Surgered Manifolds

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Annie Bart

Committee in charge:

Professor D. Darren Long, Chairperson

Professor Daryl Cooper

Professor Martin Scharlemann

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The dissertation of Annie Bart is approved

Committee Chairperson

June 1998

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Vita

September 6 1962	Born in Amsterdam, the Netherlands
1992	B.S. Mathematics, University of California, Santa Barbara
1992-1994	Teaching Assistant/Associate, University of California, Santa Barbara
1994	M.A. Mathematics, University of California, Santa Barbara
1994-1995	Graduate Research Mentorship Program
1995-1996	Teaching Assistant, University of California, Santa Barbara
1996-1997	Teaching Associate, University of California, Santa Barbara
1997-1998	Graduate Opportunity Fellowship
1998	PhD. Mathematics, University of California, Santa Barbara
1998	Assistant Professor, Saint Louis University

Publications

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Abstract

Some Results Concerning Surface Groups in Surgered Manifolds.

by Annie Bart

In this dissertation we will show that if a knot or link complement contains a closed immersed π_1 -injective quasi-fuchsian surface, then we can give a condition under which the surface will remain π_1 -injective after Dehn Surgery.

Furthermore we will give applications of this result for the Figure-8 knot complement, the Whitehead Link and the Borromean Rings. In the case of the Figure-8 knot complement we will show that there exists a surface which will remain π_1 -injective after all but thirteen surgeries. Direct computations then show that all manifolds obtained via surgery on the Figure-8 knot complement (except $(1,0)$ -surgery) contain surface groups.

Similarly we will show that for the Whitehead Link and the Borromean Rings contain a surface which will remain π_1 -injective if we exclude sixty surgeries from each link component. The result on the Whitehead Link implies that all twist knots with more than five full twist contain a closed π_1 -injective surface which will remain π_1 -injective after all but sixty surgeries.

Finally we show that if an arithmetic manifold is given so that its fundamental group is a finite index subgroup of a Bianchi group $PSL_2(\mathcal{O}_d)$ then we can find infinite families of planes in the universal cover which map down to closed totally geodesic surfaces. Furthermore we show that all these manifolds contain a π_1 -injective surface which remains injective if we exclude a finite number of surgeries from each link component.

Contents

1	Introduction	1
1.1	Background	1
1.2	Summary of Results	2
1.3	Organization	8
2	Preliminaries	10
2.1	Surgery	10
2.2	Some results concerning surface groups in manifolds	11
3	Quasi-Fuchsian Surfaces	14
3.1	Two Useful Theorems from Differential Geometry	14
3.2	π_1 -injectivity of Quasi-Fuchsian Surfaces after Dehn Surgery	15
4	Totally Geodesic surfaces	18
4.1	Arithmetic Manifolds	18
4.2	Totally Geodesic Surfaces in the Figure-Eight Knot Complement	21
4.3	The Surface S_2	23
4.4	The Combinatorial Structure of the Lifts of S_2	24
4.5	Determining the height of the maximal horosphere	27
5	Application I : the Figure-Eight Knot Complement	29
5.1	Incompressibility of S_2 after Surgery	29
5.2	The Exceptional Surgeries	31
6	Application II : the Whitehead Link	32
6.1	Surgery on Both Components of the Whitehead Link	32

6.2	Twist Knots	34
7	Application III : The Borromean Rings	38
7.1	Surgery on all Components of the Borromean Rings	38
7.2	Links and Knots obtained by filling one or two components of the Borromean Rings	40
8	Surface Groups in Some Surgered Arithmetic Manifolds.	43
9	Appendix	48
9.1	The fundamental domain of the surface S_5	48
9.2	Double curves of S_2	50

1 Introduction

1.1 Background

An important question in 3-manifold theory is the question if any homotopy equivalence between compact 3-manifolds can be induced by a homeomorphism. Due to Haken and Waldhausen, we know that Haken manifolds are determined by their fundamental group.

Recall that a manifold is **irreducible** if every embedded 2-sphere bounds a 3-ball. A surface $S \subset M$ is called **compressible** if there is an essential loop in S which bounds a disk in M . If S is not compressible (where S is not a 2-sphere or 2-disk), then S is said to be **incompressible**. A **Haken 3-manifold** is a compact, orientable, irreducible 3-manifold which contains an embedded, closed, orientable, incompressible surface S (S not a 2-sphere or 2-disk) or whose boundary is a collection of incompressible surfaces. A surface S is said to be **π_1 -injective** if $\ker(\pi_1(S) \rightarrow \pi_1(M)) = 0$. For orientable surfaces incompressible is equivalent to π_1 -injective. For a general overview of the theory of incompressible surfaces in 3-manifolds, see [AR]. We will give a short overview of the main ideas.

It is known that large classes of manifolds fail to be Haken. It has been suggested by Thurston and Waldhausen that in these manifolds we should look for immersed π_1 -injective surfaces.

We have the following conjectures :

Conjecture 1. *Let M be a closed irreducible 3-manifold with infinite first fundamental group, then M has a finite sheeted cover \widetilde{M} which is Haken.*

Conjecture 2. *Let M be a closed irreducible 3-manifold with infinite first fundamental group, then there exists an immersed closed orientable incompressible surface in M .*

Conjecture 3. *Let M be a closed irreducible 3-manifold with infinite first fundamental group, then M has a finite sheeted cover \widetilde{M} with $H_1(\widetilde{M}; \mathbb{Z})$ infinite.*

(This is equivalent to \widetilde{M} having a non-separating, closed, embedded, incompressible surface.)

Conjecture 4. *Let M be a closed irreducible 3-manifold with infinite first fundamental group, then M has a finite sheeted cover \widetilde{M} which is a closed orientable surface bundle over a circle.*

To discuss some results in this area of research, we need the following definitions. If M is an orientable hyperbolic 3-manifold, then $\pi_1(M)$ has a representation as a discrete, faithful,

torsion-free subgroup of $PSL_2(\mathbb{C})$. An immersed incompressible surface $g : S \rightarrow M$ determines a subgroup $g_*(\pi_1(S))$ of $\pi_1(M)$, and hence of $PSL_2(\mathbb{C})$. If this subgroup is conjugate into $PSL_2(\mathbb{R})$, then S is a **totally geodesic** (or **Fuchsian**) surface in M . S is said to be **quasi-Fuchsian** if the limit set of the lift of S to the universal cover is topologically a circle. If the limit set is the whole sphere at infinity then S is said to be **geometrically infinite**.

Bonahon and Thurston have shown that if a closed, orientable, hyperbolic 3-manifold M contains an π_1 -injective geometrically infinite surface, then M is finitely covered by a surface bundle over a circle, with this surface corresponding to the fiber.

Long [L] has shown that if M is a closed, orientable hyperbolic 3-manifold containing an immersed, totally geodesic surface, then M has a finite sheeted cover in which the lifted surface is embedded. Hence such manifolds are virtually Haken. Work by Marden [Ma] implies that if M is a hyperbolic 3-manifold with boundary and contains a closed π_1 -injective quasi-Fuchsian surface, then this surface will remain π_1 -injective after sufficiently large surgeries. Cooper and Long [CL] have shown recently that if M is a finite volume hyperbolic 3-manifold with a single cusp, then all but finitely many surgeries give a closed 3-manifold which contains a surface group. It does not seem to be known in these settings how to compute which surgeries would be large enough.

1.2 Summary of Results

Let M be a hyperbolic manifold which is the interior of a compact, orientable 3-manifold M^- , with ∂M^- consisting of incompressible tori. Assume that $\psi : S \rightarrow M$ is a map of a closed, orientable, π_1 -injective surface into M . S is said to contain an **accidental parabolic** if there is a loop on S homotopic into ∂M^- . There is a theorem attributed to Thurston which states that if $\psi : S \rightarrow M$ has no accidental parabolics, then for all but finitely many surgeries on any tori in ∂M^- , S remains π_1 -injective in the surgered manifold.

We will prove a version of this theorem, but our result is stated in terms of closed, immersed, quasi-Fuchsian surfaces. Furthermore, we will give a criterion to determine which surgeries will preserve injectivity. It is clear that closed quasi-Fuchsian surfaces do not contain accidental parabolics. The theorem we will prove is:

Theorem 1.1. *Let M be a complete hyperbolic 3-manifold of finite volume. Let S in M be a closed, immersed, π_1 -injective quasi-Fuchsian surface. Let Λ be the limit set of S , let $CH(\Lambda)$ be the convex hull of Λ , and let $p : \widetilde{M} \rightarrow M$ be the universal covering map.*

Let H_1, \dots, H_k be a fixed collection of horoball neighborhoods of the cusps chosen to be disjoint from $p(CH(\Lambda))$. If the p_i/q_i -surgery curves on ∂H_i have representatives which are geodesic in the induced Euclidean metric on ∂H_i and have Euclidean length $> 2\pi$, then S will remain π_1 -injective after surgery.

We will apply the main theorem to the figure-eight knot complement (denoted by M_8). We know that the figure-eight knot complement is an arithmetic manifold ([Ri],[T],[MR]). We know due to [MR] that M_8 contains infinitely many commensurability classes of closed totally geodesic surfaces. We will show that if we take a hemisphere in the universal cover (use the upper half space model) of radius \sqrt{D} centered at the origin, call it P_D , then for certain values of D this plane will map down to a closed, totally geodesic (and hence π_1 -injective) surface S_D . The theorem we will prove is :

Proposition 1.2. *Let M_8 be the figure-eight knot complement and let P_D be a plane in \mathbb{H}^3 so that $P_D \cap S_\infty^2$ is a circle of radius \sqrt{D} ($D \in \mathbb{Z}$) centered at the origin in the upper half space model. If $D \equiv 2 \pmod{3}$, then P_D projects down to a closed, immersed totally geodesic surface S_D in M_8*

We look at the surface S_2 covered by the plane P_2 , and analyze the intersection pattern in the universal cover. We know that only finitely many translates of the lifted surface will intersect one another. We will show by direct computation that for all $\epsilon > 0$ the horosphere at height $\sqrt{2} + \epsilon$ is disjoint from all the translates of the lift of the surface. Alternatively, we will also show that this has to be true because of the following lemma :

Lemma 1.3. *Let P_D be a plane in \mathbb{H}^3 which meets the sphere at infinity in a circle of radius \sqrt{D} centered at the origin and projects down to a closed surface. Let $\Gamma_d = PSL_2(\mathcal{O}_d)$. Then any translate of P_D under an element of Γ_d is a plane meeting the sphere at infinity in a circle with a radius of at most \sqrt{D} .*

We will use the information about the surface and the disjoint horoball neighborhood of the cusp when we apply our main theorem to show that there exists a surface which will remain π_1 -injective after all but thirteen surgeries.

Theorem 1.4. *The figure-eight knot contains a π_1 -injective surface, which will remain π_1 -injective after all but at most thirteen Dehn Surgeries. (These thirteen surgeries are : $\pm p/\pm q = 1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 1/2, 3/2$ or $5/2$)*

We will analyze these thirteen surgeries to show that all but one of the surgeries gives a closed manifold which contains a surface group. This gives the main theorem for the figure-eight knot complement :

Theorem 1.5. *All surgeries, except 1/0 surgery, on the figure-eight knot complement yield manifolds which contain a surface group.*

We also look at the Whitehead Link complement, call it WL . WL is known to be an arithmetic manifold. $\pi_1(WL)$ is a subgroup of $PSL_2(\mathcal{O}_1) = PSL_2(\mathbb{Z}[i])$. WL is known to contain closed totally geodesic surfaces [MR]. We look at the surface S_3 covered by the plane which meets the deck in a circle of Euclidean radius $\sqrt{3}$, and we show that this surface is closed and immersed using a lemma from [MR]. The fact that the surface is π_1 -injective follows from the fact that it is totally geodesic.

Corollary 1.6. *Let P_3 be a plane in \mathbb{H}^3 so that $P_3 \cap \mathbb{H}^3$ is a circle of radius $\sqrt{3}$. Then P_3 maps down to a closed, π_1 -injective, immersed, totally geodesic surface in the Whitehead Link.*

We analyze what happens when we perform dehn surgery on both components of the Whitehead Link and we obtain the following theorem.

Theorem 1.7. *Let WL be the Whitehead link complement and let $WL(r_1, r_2)$ be the manifold obtained by performing $r_i = p_i/q_i$ Dehn surgery on the i -th torus boundary component ($i=1,2$).*

There exists a closed π_1 -injective surface in WL which remains π_1 -injective for all but at most 60 surgeries on each link component.

(no $\pm(p_i/q_i)$ ($i = 1, 2$) belongs to the following collection of surgery slopes :

{1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 9/1, 10/1,

1/2, 3/2, 5/2, 7/2, 9/2, 1/3, 2/3, 4/3, 5/3, 7/3, 8/3,

1/4, 3/4, 5/4, 7/4, 1/5, 2/5, 3/5, 4/5})

We use this theorem to deduce that twist knots with a sufficiently large number of twists contain closed, π_1 -injective, immersed surfaces.

Corollary 1.8. *All k -twist knots where $k > 10$ contain a closed, immersed, π_1 -injective surface. Furthermore, this surface will remain π_1 -injective after all but at most 60 surgeries on the twist knot.*

We also know from [MR] that The Borromean Rings complement, BR, is an arithmetic manifold. $\pi_1(BR)$ is a subgroup of the Picard group $PSL_2(\mathbb{Z}[i])$ as is $\pi_1(WL)$. It is then easy to show that this link complement contains a closed totally geodesic surface whose lift to the universal cover is a plane P_3 , which meets the sphere at infinity in a circle of (euclidean) radius $\sqrt{3}$. We obtain a result similar to the results for the figure-eight knot and the Whitehead Link.

Theorem 1.9. *Let BR be the Borromean Rings complement and let $BR(r_1, r_2, r_3)$ be the manifold obtained by performing $r_i = p_i/q_i$ Dehn surgery on the i -th torus boundary component ($i=1,2,3$). There exists a closed π_1 -injective surface in BR which remains π_1 -injective for all but at most 60 surgeries on each link component.*

(no $\pm(p_i/q_i)$ ($i = 1, 2$) belongs to the following collection of surgery slopes :

$\{1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 9/1, 10/1,$
 $1/2, 3/2, 5/2, 7/2, 9/2, 1/3, 2/3, 4/3, 5/3, 7/3, 8/3,$
 $1/4, 3/4, 5/4, 7/4, 1/5, 2/5, 3/5, 4/5\}$)

We need some notation for the manifold we obtain when we do not perform Dehn surgery on all of the link components. Let (∞) be the label for the link component(s) we do not performing Dehn surgery on. Using this notation, we can describe the manifold we obtain by filling in two cusps of the Borromean Rings by $BR(r_1, r_2, \infty)$. Note that due to the symmetry of the Borromean Rings, there is no ambiguity. Similarly we can let $BR(r_1, \infty, \infty)$ denote the manifold obtained by filling one cusp. If we restrict our attention to $(p_1/q_1) = (1/n)$ and $(p_2/q_2) = (1/m)$, then we know the manifold thus obtained does not have torsion. Furthermore, the link components link each other trivially and the components are all unknotted. Hence the manifold obtained by Dehn surgery is a knot or a link complement. We obtain the following results :

Corollary 1.10. *Let BR be the Borromean Rings complement and let $BR(1/n, \infty, \infty)$ be the link complement obtained by performing $1/n$ Dehn surgery on one of the torus boundary components. $BR(1/n, \infty, \infty)$ contains a closed, immersed, π_1 -injective surface whenever $1/\pm n$ does not belong to the following collection of surgery slopes :*

$\{1/0, 1/1, 1/2, 1/3, 1/4, 1/5\}$

Furthermore this surface will remain π_1 -injective after all but at most 60 Dehn surgeries on each of the remaining components.

This shows that there is a family of links, which contain a closed, π_1 -injective surface.

Furthermore, we know which surgeries on the link components will leave the surface π_1 -injective.

Corollary 1.11. *Let BR be the Borromean Rings complement and let*

$BR(1/n, 1/m, \infty)$ be the knot complement obtained by performing $1/n$ and $1/m$ Dehn surgery on two of the torus boundary components. $BR(1/n, 1/m, \infty)$ contains a closed, immersed, π_1 -injective surface whenever $(1/\pm n)$ and $(1/\pm m)$ do not belong to the following collection of surgery slopes : $\{1/0, 1/1, 1/2, 1/3, 1/4, 1/5\}$

Further more this surface will remain π_1 -injective after all but at most 60 surgeries on the remaining link component.

This shows that there is a family of knots different from the twist knots, belonging to the class of 2-bridge knots, which contain a closed π_1 -injective surface. Furthermore, we know which surgeries on these knots will leave the surface π_1 -injective.

Some special arithmetic manifolds M have the property that there is a representation of the fundamental group so that $\pi_1(M) \leq PSL_2(\mathcal{O}_d)$, where \mathcal{O}_d is the ring of integers in $\mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{Z}$. $PSL_2(\mathcal{O}_d)$ is called a Bianchi group. We know via the "Cuspidal Cohomology Problem" (see [B2]) that for any arithmetic manifold d belongs to a finite set of integers, where $d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}$. Examples of knots and links for Bianchi Groups corresponding to $d = 1, 2, 3, 7, 11, 15$ and 23 can be found in [H],[MR],[NR] and [B2]. We showed that a plane of radius \sqrt{D} ($D \equiv 2 \pmod{3}$) will map down to a closed totally geodesic surface in the figure eight knot complement. This result can be extended to other arithmetic link complements whose fundamental group is a subgroup of finite index in one of the Bianchi groups mentioned above. If an arithmetic 3-manifold M is given so that $\pi_1(M) \leq PSL_2(\mathcal{O}_d)$, then we can find infinite families of planes which map down to closed, totally geodesic surfaces.

Theorem 1.12. *Suppose M is an arithmetic 3-manifold (hyperbolic) and $\pi_1(M)$ is a subgroup of finite index in $PSL_2(\mathcal{O}_d)$, then the plane of radius \sqrt{D} will project down to a closed, totally geodesic surface for the following values :*

If $d = 1$ and $D \equiv 3 \pmod{4}$.

If $d = 2$ and $D \equiv 5 \pmod{25}$.

If $d = 3$ and $D \equiv 2 \pmod{3}$.

If $d = 5$ and $D \equiv 2 \pmod{5}$.

- If $d = 6$ and $D \equiv 2 \pmod{3}$.
 If $d = 7$ and $D \equiv 3 \pmod{7}$ or $D \equiv 5 \pmod{7}$.
 If $d = 11$ and $D \equiv 2 \pmod{11}$.
 If $d = 15$ and $D \equiv 2 \pmod{3}$.
 If $d = 19$ and $D \equiv 2 \pmod{19}$.
 If $d = 23$ and $D \equiv 5 \pmod{23}$.
 If $d = 31$ and $D \equiv 3 \pmod{31}$.
 If $d = 39$ and $D \equiv 2 \pmod{3}$.
 If $d = 47$ and $D \equiv 5 \pmod{47}$.
 If $d = 71$ and $D \equiv 7 \pmod{71}$.

For a knot or link complement M , where $\pi_1(M) \leq PSL_2(\mathcal{O}_d)$, we can also compute what the minimal length L of a longitude has to be if we normalize the length of the meridian. This is enough to show that the following theorem holds:

Theorem 1.13. *Suppose M is an arithmetic 3-manifold (hyperbolic) and $\pi_1(M)$ is a subgroup of finite index in $PSL_2(\mathcal{O}_d)$. Let L be the minimal length of the longitude after normalizing the meridian to have length 1. Let D be chosen so that the surface covered by the plane of a given radius \sqrt{D} is a closed totally geodesic surface. This surface will remain π_1 -injective if for the p/q surgeries on each component the surgery condition holds as shown in the table. We also give an upperbound on the number of bad surgeries (where the surface might compress).*

d	D	L	surgery condition	# bad surgeries
1	3	$\sqrt{2}$	$p^2 + 2q^2 > 3(2\pi)^2$	83
2	5	$\sqrt{3}$	$p^2 + 3q^2 > 5(2\pi)^2$	111
3	2	1	$p^2 + q^2 > 2(2\pi)^2$	77
5	2	$\sqrt{6}$	$p^2 + 6q^2 > 2(2\pi)^2$	33
6	2	$\sqrt{7}$	$p^2 + 7q^2 > 2(2\pi)^2$	31
7	3	$\sqrt{2}$	$p^2 + 2q^2 > 3(2\pi)^2$	83
11	2	$\sqrt{3}$	$p^2 + 3q^2 > 2(2\pi)^2$	45
15	2	2	$p^2 + 4q^2 > 2(2\pi)^2$	39
19	2	$\sqrt{5}$	$p^2 + 5q^2 > 2(2\pi)^2$	35
23	5	$\sqrt{6}$	$p^2 + 6q^2 > 5(2\pi)^2$	79
31	3	$\sqrt{8}$	$p^2 + 8q^2 > 3(2\pi)^2$	41
39	2	$\sqrt{10}$	$p^2 + 10q^2 > 2(2\pi)^2$	25
47	5	$\sqrt{12}$	$p^2 + 12q^2 > 5(2\pi)^2$	55
71	7	$\sqrt{18}$	$p^2 + 18q^2 > 7(2\pi)^2$	63

This will allow us to show the following :

Theorem 1.14. *Let L be an arithmetic link whose fundamental group is a subgroup of finite index in a Bianchi group. If L' is a knot or link complement obtained via $1/n_i$ surgery on one or more unknotted components of L and none of the $1/n_i$ surgeries are bad, then L' will contain a closed π_1 -injective surface. Furthermore this surface which will remain π_1 -injective after all but a finite number of surgeries on each component of L' . The number of bad surgeries is determined by the Bianchi group associated with the original arithmetic link complement.*

1.3 Organization

In section 2 we will review Dehn surgery, in particular Dehn surgery on the figure-eight knot complement. We will give an overview of what is already known about surfaces in 3-manifolds, surfaces in the figure-eight knot complement and surfaces in twist knot complements. We will also give a summary of what is known about \mathbb{Z} -representability of manifolds obtained by surgery on the figure-eight knot complement.

In section 3 we will review some facts from differential geometry. We will need the 2π -theorem by Thurston and Gromov and Hadamard's theorem to prove our main result. We will then state and prove the main theorem about incompressibility of quasi-Fuchsian surfaces after surgery.

In section 4 we will review some general theory about arithmetic manifolds, which we will use to prove that certain planes in the universal cover map down to closed, totally geodesic surfaces in the figure-eight knot complement M_8 . We will then focus our attention on one of these surfaces and analyze how the lifts of this surface meet one another. In the last subsection, we will give a general argument why the translates of a lift of a closed totally geodesic surface never move into a fixed cusp. We will also show how to determine this cusp.

In section 5 we will use the information from section 4 and the theorem from section 3 to prove that one of the totally geodesic surfaces remains π_1 -injective after all but thirteen surgeries. This theorem, combined with direct computations and general knowledge about Seifert Fibered spaces will give us the main theorem about surgery on the figure-eight knot complement.

In section 6 we use the Wirtinger presentation of the fundamental group of the Whitehead link complement to find a representation of the fundamental group into $PSL_2(\mathbb{Z}[i])$. We show that the plane which traces out a circle of euclidean radius $\sqrt{3}$ in the sphere at

infinity maps down to a closed, immersed, totally geodesic surface in the Whitehead link complement. We then show that this surface remains π_1 -injective after Dehn surgery on both components of the link if we exclude 60 surgery slopes. A corollary of this theorem is that twist knots with more than 5 full twists contain a closed, immersed, π_1 -injective surface. Furthermore we show that this surface remains π_1 -injective after all but 60 surgeries on the twist knot.

In section 7 we find the Wirtinger presentation of the Borromean Rings. Results from a paper by Hilden, Lozano and Montesinos give a representation of the fundamental group into $PSL_2(\mathbb{Z}[i])$. This link complement also contains a totally geodesic surface. Using the same techniques as were used in previous sections we then show that the surface remains π_1 -injective after all but a finite number of surgeries. We also consider what happens when we fill only one or two components of the Borromean Rings. We show that there is an infinite family of two component links which contain closed π_1 -injective surfaces, and we show that there is another infinite family of 2-bridge knots which contain closed π_1 -injective surfaces. Finally we show which surgeries on these links and knots will leave the closed π_1 -injective surface π_1 -injective.

In section 8 we show that if an arithmetic manifold is given so that its fundamental group is a finite index subgroup of a Bianchi group $PSL_2(\mathcal{O}_d)$ then we can find infinite families of planes in the universal cover which map down to closed totally geodesic surfaces. Furthermore we show that all these manifolds contain a π_1 -injective surface which remains injective if we exclude a finite number of surgeries from each link component. It can be shown that the number of excluded surgeries for each of the link components of these knot and link complements is bounded above by 111.

2 Preliminaries

2.1 Surgery

It is a well known general fact that compact 3-manifolds can be triangulated. Suppose we have a complete manifold M with a given triangulation. If we deform the triangulation of the manifold M , we no longer have a complete structure. Completing the structure corresponds to doing surgery on the original manifold M . The following discussion of surgery is a combination of what was done in Benedetti and Petronio [BP] and Thurston's Notes [T].

To complete the space, we clearly need to add something near a vertex. Let M be a hyperbolic manifold obtained by gluing a polyhedron P with some vertices at infinity. Let K be the complex obtained by adding the vertices at infinity. The completion \overline{M} is obtained by completing a deleted neighborhood $N(v)$ of each ideal vertex v in K and gluing these completed neighborhoods $\overline{N(v)}$ back into M .

If we denote the completion of $N(v)$ by $\overline{N(v)}$, then a cross-section of $\overline{N(v)}$ perpendicular to the added circle is a cone C_θ . When $\theta = 2\pi$, C_θ is non-singular we see that the completion of M is a hyperbolic manifold. So we want some conditions which tell us when $\theta = 2\pi$.

Consider any 3-manifold which is the interior of a compact manifold M^- which has one torus boundary component T . Let λ and μ be the standard generators of $\pi_1(T)$. Let $M(p/q)$ denote the manifold obtained by gluing a solid torus to M^- so that the meridian of this torus is mapped to $p\mu + q\lambda$. If (p/q) is replaced by ∞ , then nothing is glued in. Hence $M = M_\infty$.

The generalized Dehn surgery invariants (p, q) for M^- are solutions to the equation
:

$$p\tilde{H}(\mu) + q\tilde{H}(\lambda) = \text{rotation by } \pm 2\pi, (p, q) = 1$$

(or $p/q = \infty$ if M is complete near T)

Alternatively we can think of surgery on a knot or link complement as truncating the cusp, so that we obtain a compact manifold with torus boundary components. A (p, q) Dehn filling on one of the torus boundary components then corresponds to taking a disc D and attaching a neighborhood of the disc, $D \times I$, along its boundary $\partial D \times I$ to a annulus neighborhood of a (p, q) torus curve. Note that p and q have to be relatively prime. After attaching $D \times I$ in this manner we obtain a manifold with a spherical boundary component,

which we cap off to obtain the Dehn filling.

In the particular case of the figure eight-knot complement we know exactly which fillings result in a hyperbolic manifold. In particular we have :

Theorem 2.1. *(Thurston) Every manifold obtained by Dehn surgery along the figure-eight knot has a hyperbolic structure, except the six manifolds :*

$M_8(\pm p/\pm q)$ where $p/q = 1/0, 0/1, 1/1, 2/1, 3/1$ or $4/1$

We have a similar result for surgery on one component of the Whitehead link complement. This result can be found in [NR].

Theorem 2.2. *Every manifold obtained by Dehn surgery on one component of the Whitehead link complement WL has a hyperbolic structure, except the six manifolds $WL_{\pm(p/q)}$, where $p/q = 1/0, 0/1, -1/1, -2/1, -3/1$ or $-4/1$*

2.2 Some results concerning surface groups in manifolds

In this dissertation we will investigate some properties of the figure-eight knot complement and manifolds obtained from it via surgery.

General facts about surfaces in the figure-eight knot complement can be found in a paper by Aitchinson, Matsumoto and Rubinstein [AMR]. Suppose the figure-eight knot complement is given the triangulation as described in Thurston's Notes [T]. This triangulation induces a triangulation on any normal surface immersed in the knot complement. The induced triangulation consists of triangles and squares.

Lemma 2.3. *Suppose S is a closed normal surface immersed in M_8 , Then*

1. *Each vertex of S is of degree 6 and*
2. *Every vertex has an even number (possibly 0) of squares around it.*

Lemma 2.4. *There are exactly eight possible vertex types in regular normal surfaces S in M_8*

Proof: By the previous lemma ,the number q of squares around a given vertex is 0, 2, 4 or 6.

If $q = 0$, then the type is $TTTTTT$

If $q = 2$, then there are three possible types : $QQTTTT, QTQTTT$ and $QTTQTT$.

If $q = 4$, then there are three possible types : $QQQQTT, QQQTQT$ and $QQTQQT$.

And $q = 6$ gives vertex type $QQQQQQ$.

□

Theorem 2.5. *Suppose S is a normal surface immersed in M_8 . S is compressible if:*

1. *There is an infinite row of squares in the universal cover \tilde{S} of S .*
2. *S is a homogeneous surface of vertex type $QQTTTT$*
3. *S is a homogeneous surface of vertex type $QTTQTT$*
4. *S is a homogeneous surface of vertex type $QQQQTT$*
5. *S is a homogeneous surface of vertex type $QQTQQT$*
6. *S is a homogeneous surface of vertex type $QQQQQQ$*

A lot is already known about manifolds obtained by surgery on the figure-eight knot. Much of this information can be found in papers by M. Baker [B] and J. Przytycki [P].

If M has finite cover N with $\text{Rank } H_1(N; \mathbb{Z}) > 0$, then M is called **virtually \mathbb{Z} -representable** or **has virtually positive first Betti number**. With irreducibility this implies that M is virtually Haken.

If we denote the manifold obtained via (p/q) surgery on the figure-eight knot complement by $M_{(p/q)}$, then we have the following results:

Theorem 2.6. *$M_{(p/q)}$ is known to have a virtually \mathbb{Z} -representable fundamental group if :*

- (i) $q \equiv \pm 2p \pmod{7}$ (Hempel, Nicas)
- (ii) $q \equiv \pm p \pmod{13}$ (Hempel, Nicas)
- (iii) $p \equiv \pm 0 \pmod{4}$ and $\frac{p}{q} \neq \pm 8$ (Kojima and Long)

Theorem 2.7. (Baker)

- (A) $M_{(3p/q)}$ has a virtually \mathbb{Z} -representable fundamental group if $|q| \notin \{p-1, p+1\}$.
- (B) $M_{(4p/q)}$ has a virtually \mathbb{Z} -representable fundamental group.
- (C) $M_{(2p/q)}$ has a virtually \mathbb{Z} -representable fundamental group if $q \equiv \pm 7p \pmod{15}$.

Theorem 2.8. (Przytycki)

Let M_8 be the figure eight knot complement. Let p, q be coprime numbers such that either :

$0 < |p| \leq (|q| - 5)/4$ or

$|p| = (|q| - 3)/4$ or

$p/q \in \{1/0, \pm 1/1, \pm 1/2, \pm 1/3, \pm 1/4\}$

Then $M(p/q)$ is virtually Haken.

Furthermore, if $p/q \in \{\pm 1/1, \pm 1/2, \pm 1/3\}$, then $M(p/q)$ is a Seifert fibered manifold.

If $p/q \in \{1/0, \pm 1/4\}$, then $M(p/q)$ is a Haken manifold.

In the other cases considered in this theorem $M(p/q)$ is neither Haken nor a Seifert fibered manifold, but a virtually Haken, hyperbolic manifold.

We will also investigate surfaces in the Whitehead link and in twist knots. We will show that if the twist-knot has more than 5 full twists, then it must contain a closed, π_1 -injective, immersed surface. This is interesting in the light of the following theorem which appears in [R]:

Theorem 2.9. *There exist infinitely many twist knots which contain no closed totally geodesic surface and exactly one commensurability class of non-closed totally geodesic surface.*

3 Quasi-Fuchsian Surfaces

3.1 Two Useful Theorems from Differential Geometry

Let M_K denote a knot complement, and let M_K^- denote the compact manifold obtained by truncating the cusp. Then the surgered manifold M_σ can be thought of as $M_\sigma = M_K^- \cup_h V$, where V is a solid torus and $h : \partial V \rightarrow \partial M_K^-$ is the boundary homeomorphism which takes the meridian of ∂V to $p\mu + q\lambda$ (where $(p, q) = 1$). If the surgery curve is long enough, we are guaranteed a metric of negative curvature on the resulting manifold. This is known as the 2π theorem of Thurston and Gromov, a proof of which is given in Bleiler and Hodgson [BH].

Theorem 3.1. *(The 2π -theorem) Let M be a complete hyperbolic 3-manifold of finite volume and H_1, \dots, H_k disjoint horoball neighborhoods of the cusps of M . Suppose r_i is a slope on ∂H_i represented by a geodesic α_i with length in the Euclidean metric satisfying $\text{length}(\alpha_i) > 2\pi$, for each $i = 1, \dots, k$. Then $M(r_1, \dots, r_k)$ has a metric of negative curvature.*

To find the length of a (p/q) -surgery curve on the boundary torus of a compact manifold, we lift a geodesic representative of the curve to the universal cover. The boundary torus will lift to a horosphere at height h . The geodesic representative α of the (p/q) -curve in the boundary torus will lift to a straight line in the horosphere. The meridian μ and the longitude λ also lift to this horosphere, call the lifts $\tilde{\mu}$ and $\tilde{\lambda}$, and we can think of these lifts as vectors in the horosphere which cover the meridian and the longitude in a one-to-one fashion. The line segment L going from $(0, 0)$ to $(p\mu, q\lambda)$ will cover the surgery curve in the boundary torus once. The horosphere has a scaled version of the Euclidean metric and we say that the length of L is the length of the (p/q) surgery curve. In particular, we can arrange it so that the length L is given by the following formula:

$$L = \sqrt{p^2 \|\tilde{\mu}\|^2 + q^2 \|\tilde{\lambda}\|^2} / h$$

The proof of the 2π theorem is based on the following lemma, which is worth mentioning. We can define a metric on the surgered manifold $M_\sigma = M_K^- \cup V$, where we have the hyperbolic metric on M_K^- and extend this metric to a metric of negative curvature on the solid torus V . This idea will be an important ingredient of the proof of the main theorem in the next section.

Lemma 3.2. *Let V be a solid torus supplied with a hyperbolic metric near its boundary so that ∂V is the quotient of a horosphere. Then the metric near the boundary can be*

extended to a negatively curved metric on V provided that the length of the Euclidean geodesic representing the meridian curve on ∂V is at least 2π .

To prove our result, we also need the the following theorem which can be found in doCarmo (see [DC2]).

Theorem 3.3. (Hadamard) *Let M be a complete Riemannian manifold, simply connected, with sectional curvature $K(p, \Sigma) \leq 0$, for all $p \in M$ and for all sections $\Sigma \subset T_p(M)$. Then M is diffeomorphic to \mathbb{R}^n , $n = \dim M$; more precisely $\exp_p : T_p(M) \rightarrow M$ is a diffeomorphism*

Recall that **geodesics** are characterized by the fact that they are locally distance minimizing. The last part of the statement of Hadamard's theorem immediately implies the following corollary:

Corollary 3.4. *Let M be a complete Riemannian manifold. Suppose all sectional curvatures are less than or equal to 0 and $\pi_1(M) = 1$, then there are no closed geodesics.*

3.2 π_1 -injectivity of Quasi-Fuchsian Surfaces after Dehn Surgery

Let Γ be any discrete group of orientation preserving isometries of \mathbb{H}^n . If $x \in \mathbb{H}^n$, then **the limit set** $\Lambda_\Gamma \subset S_\infty^{n-1}$ is defined to be the set of accumulation points of the orbits Γ_x of x . Γ is said to be **an elementary group** if the limit set consists of 0,1 or 2 points. From now on we will always assume that Γ is non-elementary.

Now let S be a quasi-Fuchsian surface in a hyperbolic manifold. Let \tilde{S} be the lift of S to the universal cover \tilde{M} of M . Suppose \tilde{S} meets the sphere at infinity in a set K . Define $CH(K)$ to be the convex hull of K . Where the hyperbolic convex hull is the intersection of all hyperbolic half spaces in \mathbb{H}^3 whose intersection with the sphere at infinity contain K . For more detail, see chapter 8 of Thurston's Notes [T].

We may assume that the lift \tilde{S} of a π_1 -injective quasi-Fuchsian surface S is contained in the convex hull. The frontier of the convex hull consists of geodesic segments. Hence this frontier will map down to a surface in the manifold, which is clearly homotopic to the surface we started with. Hence we might as well assume that the lift of the surface to the universal cover is contained in the convex hull.

Lemma 3.5. *Let M be a complete hyperbolic 3-manifold of finite volume. Let S in M be a closed, immersed, π_1 -injective quasi-Fuchsian surface. Let Λ be the limit set of $\pi_1(S)$.*

Let $CH(\Lambda)$ be the convex hull of Λ and let $p : \widetilde{M} \rightarrow M$ be the universal covering map. Then $p(CH(\Lambda))$ is compact.

Proof: S is a closed quasi-Fuchsian surface, hence its fundamental group does not contain any parabolics. This implies that $CH(\Lambda)/\pi_1(S)$ must be compact. Note that $p : \mathbb{H}^3 \rightarrow \mathbb{H}^3/\pi_1(M)$ is a local isometry. Hence so is p restricted to $CH(\Lambda)$. The map $g : CH(\Lambda) \rightarrow CH(\Lambda)/\pi_1(S)$ is a covering map and hence a local isometry. Hence we can define a map $\phi : CH(\Lambda)/\pi_1(S) \rightarrow M$ so that this map is a local isometry. Because $CH(\Lambda)/\pi_1(S)$ is compact, we see that any point x in $p(CH(\Lambda))$ is a bounded distance away from S . This implies that $p(CH(\Lambda))$ must be compact. \square

Corollary 3.6. *Let M be a complete hyperbolic 3-manifold of finite volume. Let S in M be a closed, immersed, π_1 -injective quasi-Fuchsian surface. Let Λ be the limit set of $\pi_1(S)$. Let $CH(\Lambda)$ be the convex hull of Λ and let $p : \widetilde{M} \rightarrow M$ be the universal covering map. Then we can choose horoball neighborhoods of the cusp which do not meet $p(CH(\Lambda))$.*

This lemma shows we can truncate the manifold M , so that the boundary tori of the truncated manifold do not intersect the projection of $CH(\Gamma)$.

Theorem 3.7. *Let M be a complete hyperbolic 3-manifold of finite volume. Let S in M be a closed, immersed, π_1 -injective quasi-Fuchsian surface. Let Λ be the limit set of $\pi_1(S)$, let $CH(\Lambda)$ be the convex hull of Λ , and let $p : \widetilde{M} \rightarrow M$ be the universal covering map. Let H_1, \dots, H_k be a fixed collection of horoball neighborhoods of the cusps chosen to be disjoint from $p(CH(\Lambda))$. If the p_i/q_i -surgery curves on ∂H_i have representatives which are geodesic in the induced Euclidean metric on ∂H_i and have Euclidean length $> 2\pi$, then S will remain π_1 -injective after surgery.*

Proof: As in lemma 3.5 we can define a map $\phi : CH(\Lambda)/\pi_1(S) \rightarrow M$ so that this map is a local isometry. Let $X = CH(\Lambda)/\pi_1(S)$. By hypothesis $p(CH(\Lambda))$ is contained in M^- . Hence there is a map $\phi^- : X \rightarrow M^-$ so that ϕ^- is a local isometry. Let $r = p_1/q_1, \dots, p_k/q_k$ denote the surgery curves. There is an inclusion map $i : M^- \rightarrow M(r)$. The 2π -theorem implies that we can put a metric on $M(r)$ so that $\phi_r : X \rightarrow M(r)$, where $\phi_r = i \cdot \phi^-$, is also a local isometry. The map ϕ_r is π_1 -injective. To see this suppose that $\gamma \in \ker(\phi_r)$. Then because of convexity γ is freely homotopic to a geodesic in X . So assume that γ is a geodesic. But then $\phi_r(\gamma)$ is a null homotopic geodesic in $M(r)$. This contradicts the corollary to Hadamard's theorem. Hence ϕ_r is π_1 -injective and hence S is a π_1 -injective

surface in $M(r)$. □

In the special case of a totally geodesic surface in a knot or link complement we obtain an easy corollary. We will include an alternative proof.

Corollary 3.8. *Let M be a hyperbolic knot or link complement. Let S in M be a closed, immersed, π_1 -injective totally geodesic surface. If the (p/q) -surgery curves lie on boundary tori disjoint from the surface S and lift to a horosphere where they have Euclidean length $> 2\pi$, then S will remain π_1 -injective after surgery.*

Proof: The limit set Γ is a round circle. Hence the convex hull is just the lift of the totally geodesic surface. Suppose there is a compressing disk D , then $\partial D = \alpha$ is an essential curve in S . But α is freely homotopic to a curve α^* , where α^* is a geodesic in S . But S is totally geodesic, hence α^* is a geodesic in M^- . By defining the appropriate metric on the surgered manifold $M_\sigma = M^- \cup_h (\cup_{i=1}^n V_i)$ (where V_i are solid tori), α^* is a geodesic in M_σ . But clearly α^* bounds a disk. Hence α^* lifts to the universal cover. This contradicts the corollary to Hadamard's theorem. Hence S remains π_1 -injective after surgery on sufficiently large surgery curves. □

4 Totally Geodesic surfaces

4.1 Arithmetic Manifolds

An interesting class of manifolds are arithmetic manifolds, because some of them can be shown to contain totally geodesic surfaces which are automatically π_1 -injective. Background can be found in [M], [MR], [R] and [V]. We will now give a summary of the most important ideas. To define what an arithmetic manifold is, we first need to define what an order in a quaternion algebra is.

K , a subfield of the Complex numbers, is a **numberfield** if $[K : \mathbb{Q}] < \infty$. \mathcal{A} is a **quaternion algebra** over a numberfield K , with basis $\{1, i, j, ij\}$, if \mathcal{A} is a 4-dimensional vectorspace so that $i^2 = a, j^2 = b, ij = ji$, where $a, b \in K - \{0\}$.

We say that an element x in the quaternion algebra \mathcal{A} is an **integer** if $tr(x) = x + \bar{x} \in \mathbb{Z}$, and $n(x) = x\bar{x} \in \mathbb{Z}$. Where given that $x = x_0 + x_1i + x_2j + x_3ij$, and we defined $\bar{x} = x_0 - x_1i - x_2j - x_3ij$.

\mathcal{O} in \mathcal{A} is an **order** if

1. \mathcal{O} is a finitely generated subring of \mathcal{A} consisting of integers.
2. \mathcal{O} contains a K -basis for \mathcal{A} .
3. \mathcal{O} contains R_K (the ring of integers of K).

Define $\mathcal{O}^1 = \{x \in \mathcal{O} \mid n(x) = 1\}$.

A **Kleinian group** is a discrete subgroup of $PSL_2(\mathbb{C})$ and a **Fuchsian group** is a Kleinian group which stabilizes a circle or straight line \mathcal{C} in \mathbb{C} and preserves the components of $\mathbb{C} \setminus \mathcal{C}$. Let Γ be a finite covolume group. Define ρ to be an isomorphism of \mathcal{A} into $M_2(\mathbb{C})$, and let P be the projection $P : GL_2(\mathbb{C}) \rightarrow PGL_2(\mathbb{C})$.

Any $\Gamma \leq P\rho\mathcal{O}^1$ is said to be **derived from a quaternion algebra**. Any Kleinian group commensurable with $\Gamma \leq P\rho\mathcal{O}^1$ is called an **arithmetic Kleinian group**.

H and K , groups in X , are **commensurable** if $[H : H \cap K]$ and $[K : H \cap K]$ are both finite.

If S is a Fuchsian group stabilizing \mathcal{C} , then S is **non-elementary** if its limit set on \mathcal{C} consists of more than two points.

Example 1. An Arithmetic Fuchsian Group

$$\text{Let } \mathcal{A} = \left(\frac{-3, 2}{\mathbb{Q}} \right) = \{a_0 + a_1i + a_2j + a_3ij \mid a_i \in \mathbb{Q}, i^2 = -3, j^2 = 2, ij = -ji\}$$

be a quaternion algebra.

$\mathcal{O} = \mathbb{Z}[1, i, j, ij]$ is an order in \mathcal{A} .

(\mathcal{O} is a finitely generated subring of \mathcal{A} , and consists of integers : $x + \bar{x} = 2a_0 \in \mathbb{Z}$ and $x\bar{x} = a_0^2 - 3a_1^2 + 2a_2^2 - 6a_3^2 \in \mathbb{Z}$; \mathcal{O} contains a \mathbb{Q} basis, and \mathcal{O} contains $R_{\mathbb{Q}} = \mathbb{Z}$.)

Now consider $\mathcal{A} = \left(\frac{-3, 2}{\mathbb{Q}} \right)$. \mathcal{A} clearly splits over $\mathbb{Q}(\sqrt{-3})$. Hence we obtain the following representation

$$\rho : \mathcal{A} = \left(\frac{-3, 2}{\mathbb{Q}} \right) \longrightarrow M_2(\mathbb{Q}(\sqrt{-3}))$$

defined by

$$\rho : a_0 + a_1i + a_2j + a_3ij \longrightarrow \begin{pmatrix} a_0 + a_1\sqrt{-3} & 2(a_2 + a_3\sqrt{-3}) \\ a_2 - a_3\sqrt{-3} & a_0 - a_1\sqrt{-3} \end{pmatrix}$$

If we now restrict the map ρ to \mathcal{O}^1 , we see that

$$P\rho\mathcal{O}^1 = \left\{ \begin{pmatrix} \alpha & 2\beta \\ \beta & \alpha \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}[\omega]) \mid |\alpha|^2 - 2|\beta|^2 = 1 \right\}$$

This group stabilizes the circle $|z|^2 = 2$ in S_{∞}^2 and hence is an arithmetic Fuchsian group.

□

General results about Kleinian groups and their Fuchsian subgroups can be found in [R], [MR] and [M]. The results of interest to us are the conditions under which co-compact Kleinian arithmetic groups contain co-compact Fuchsian subgroups.

Theorem 4.1. (Reid) *Let Γ be a co-compact arithmetic Kleinian group. If Γ contains a non-elementary Fuchsian subgroup then Γ contains infinitely many commensurability classes (up to conjugacy in $\text{PSL}_2\mathbb{C}$) of co-compact (necessarily arithmetic) Fuchsian subgroups.*

Maclachlan and Reid [MR] have shown that the figure-eight knot complement, the Borromean Rings complement and the Whitehead Link complements contain infinitely many incommensurable, immersed, closed, totally geodesic surfaces.

To prove some of the results we need, we have to give some definitions and state some known theorems. The following material comes from the paper written by Maclachlan and

Reid [MR] and from lecture notes from a class taught by Darren Long [LL] at the University of California at Santa Barbara.

Let \mathcal{O}_d be the ring of integers in $\mathbb{Q}(\sqrt{-d})$, and let Γ_d denote $PSL_2(\mathcal{O}_d)$. Suppose that $PSL_2(\mathcal{O}_d)$ contains a non-elementary Fuchsian subgroup which stabilizes a circle or straight line \mathcal{C} in \mathbb{C} (the sphere at infinity in the upper half space model.) It can be shown (see [M]) that the circle (line) \mathcal{C} has an equation of the form

$$a|z|^2 + Bz + \bar{B}\bar{z} + c = 0, \text{ where } a, c \in \mathbb{Z} \text{ and } B \in \mathcal{O}_d$$

Let $B = \frac{1}{2}(b_1 + b_2\sqrt{-d})$ with $b_i \in \mathbb{Z} (i = 1, 2)$ and $b_1 \equiv b_2 \pmod{2}$ (and $\equiv 0 \pmod{2}$ unless $d \equiv -1 \pmod{4}$). The triple (a, B, c) is called **primitive** if

$$\gcd(a, \frac{b_1}{2}, \frac{b_2}{2}, c) = 1 \text{ for } b_1 \equiv b_2 \equiv 0 \pmod{2}$$

$$\gcd(a, b_1, b_2, c) = 1 \text{ for } b_1 \equiv b_2 \equiv 1 \pmod{2}$$

The **discriminant** of \mathcal{C} is defined to be

$$D = \frac{1}{4}(b_1^2 + db_2^2) - ac$$

We have the following theorem (see Maclachlan [M])

Theorem 4.2. (Maclachlan) *Every maximal non-elementary Fuchsian subgroup of $PSL_2(\mathcal{O}_d)$ is an arithmetic Fuchsian group arising from some quaternion algebra $\left(\frac{-d, D}{\mathbb{Q}}\right)$. In particular they have finite covolume.*

In otherwords, this means that the quaternion algebra associated to the Fuchsian subgroup of Γ_d of discriminant D is isomorphic to $\left(\frac{-d, D}{\mathbb{Q}}\right)$

Proposition 4.3. (Maclachlan) *Let S be a totally geodesic surface contained in an arithmetic manifold M whose fundamental group, $\pi_1(M)$, is contained in $PSL_2(\mathcal{O}_d)$. A lift of S meets the sphere at infinity in a circle \mathcal{C} . The stabilizer of \mathcal{C} is :*

$$\text{Stab}(\mathcal{C}) = \left\{ \begin{pmatrix} \alpha & D\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in PSL_2(\mathcal{O}_d) \mid |\alpha|^2 - D|\beta|^2 = 1 \right\}$$

where D is the discriminant of the circle at infinity.

We need to know if a group Γ is co-compact or not. To get an answer to this question we need to define $A(\Gamma) = \{\sum_{finite} a_i \gamma_i \mid \gamma_i \in \Gamma, a_i \in \mathbb{Q}(tr\Gamma)\}$, where $\mathbb{Q}(tr\Gamma) = \mathbb{Q}(tr\gamma \mid \gamma \in \Gamma)$ is the **Trace Field**

Theorem 4.4. *Let Γ be an arithmetic Kleinian group. Then Γ is non-co-compact if and only if $\mathcal{A} = A(\Gamma^{(2)})$ is isomorphic to $M_2(\mathbb{Q}(\sqrt{-d}))$. (Where $\Gamma^{(2)} = \{\gamma^2 \mid \gamma \in \Gamma\}$)*

Sketch of proof : If \mathbb{H}^3/Γ is compact, then there is a lowerbound on the distance that Γ moves a point in \mathbb{H}^3 . Hence Γ contains no parabolics. But $SL_2(R_{\sqrt{-d}})$ does.

Conversely if Γ is arithmetic and not co-compact, then Γ contains parabolics. Hence $\Gamma^{(2)}$ contains parabolics. It can be shown that this implies that $\mathcal{A} = M_2(K)$ where K is a field. It can further be shown that K is a dimension 2 extension over \mathbb{Q} , so that $K = \mathbb{Q}(\sqrt{-d})$ \square

Theorem 4.5. *Let \mathcal{A} be a quaternion algebra over K . Then $n_{\mathcal{A}/K}$ is isotropic ($\exists x \neq 0 \in \mathcal{A}$, so that $n(x) = 0$) if and only if \mathcal{A} is isomorphic to $M_2(K)$.*

Sketch of proof: If \mathcal{A} is isomorphic to $M_2(K)$ then there are non-invertible matrices hence $n_{\mathcal{A}/K}$ is isotropic.

Conversely suppose I is a right ideal in \mathcal{A} . Then in particular I is a vectorsubspace of \mathcal{A} . This implies the descending chain condition on right ideals (every descending chain terminates). Then by a theorem by Wedderburn we see that \mathcal{A} is isomorphic to a matrix algebra $M_k(\Delta)$, where Δ is a division ring and $k \geq 1$. The normform $n_{\mathcal{A}/K}$ is isotropic, hence $k \geq 2$. But $\Delta = K$ by equality of centers, so that $k = 2$, because \mathcal{A} has dimension 4 over its center. \square

4.2 Totally Geodesic Surfaces in the Figure-Eight Knot Complement

Let L be the figure-eight knot and let M_8 be $S^3 \setminus N(L)$.

Then $\pi_1(M_8) = \langle a, b \mid b^{-1}aba^{-1}b = ab^{-1}aba^{-1} \rangle$, where we can choose

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 0 \\ \omega + 1 & 1 \end{pmatrix}$$

with $\omega^3 = 1, \omega \neq 1$ (for a reference see [Ri]). A description of M_8 is given in Thurston's notes [T]. M_8 can be thought of as union of two tetrahedra, glued together according to the gluing conditions shown in Figure 1.

Hyperbolic upper half space \mathbb{H}^3 can be triangulated by tetrahedra. A totally geodesic surface in M_8 lifts to a plane in \mathbb{H}^3 . Suppose S_D lifts to a plane P_D , where $P_D \cap S_\infty^2$ is a round circle of radius \sqrt{D} , centered at the origin. We have a list of possible surfaces to work with, via the following proposition :

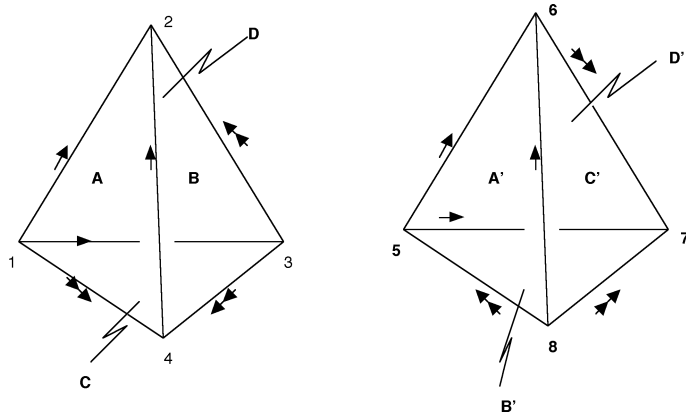


Figure 1: Triangulation of figure-8-knot complement (The tetrahedra are projected down to a plane)

Theorem 4.6. *Let M_8 be the figure-eight knot complement and let P_D be a plane in \mathbb{H}^3 so that $P_D \cap \mathbb{H}^3$ is a circle of radius \sqrt{D} ($D \in \mathbb{Z}$) centered at the origin in the upper half space model. If $D \equiv 2 \pmod{3}$, then P_D projects down to a closed, immersed totally geodesic surface S_D in M_8*

Proof: D is a square free number. Hence the equation $|\alpha|^2 - D|\beta|^2 = 1$ has infinitely many integral solutions. (This is Pell's Equation.) This implies that $Stab(S_D)$ is non-elementary. From [M,MR] it then follows that $Stab(S_D)$ is finite covolume. $Stab(S_D)$ is a maximal subgroup of $PSL_2(\mathcal{O}_d)$ and $\pi_1(M)$ is a subgroup of finite index, hence the fundamental group of the surface will be of finite index in $Stab(S_D)$. Hence to show that the surface is compact, it suffices to show that $Stab(S_D)$ is cocompact. By the previous theorem the quaternion algebra \mathcal{A} associated to the Fuchsian subgroup of Γ_3 of discriminant D is isomorphic to $\left(\frac{-3, D}{\mathbb{Q}}\right)$. Hence a typical element x from \mathcal{A} has the following general form: $x = x_0 + x_1i + x_2j + x_3ij$, with $x_i \in \mathbb{Q}$, $i^2 = -3$, $j^2 = D$ and $ij = -ji$. The norm of x ,

$n(x)$, is given by $n(x) = x\bar{x}$. Note that when we consider the equation $x\bar{x} = 0$ we actually can assume that $x_i \in \mathbb{Z}$ (multiply through by the appropriate least common multiple of the denominators of the x_i). It is easy to check that the normform $n_{\mathcal{A}/K}$ is not isotropic : Suppose $n_{\mathcal{A}/K}(x) = x_0^2 + 3x_1^2 - Dx_2^2 - 3Dx_3^2 = 0$ and $x \neq 0$

$D \equiv 2 \pmod{3}$ implies that $x_0^2 - 2x_2^2 \equiv 0 \pmod{3}$. Hence $x_0^2 \equiv x_2^2 \equiv 0 \pmod{3}$.

This implies that $x_0^2 = 9\bar{x}_0^2$ and $x_2^2 = 9\bar{x}_2^2$ and we obtain obtain the expression $3\bar{x}_0^2 + x_1^2 - 3D\bar{x}_2^2 - Dx_3^2 = 0$. This now implies that $x_1^2 - 2x_3^2 \equiv 0 \pmod{3}$. By a similar argument we get $\bar{x}_0^2 + 3\bar{x}_1^2 - D\bar{x}_2^2 - 3D\bar{x}_3^2 = 0$. Clearly we eventually reach a contradiction. Hence $n_{\mathcal{A}/K}$ is not isotropic and hence by theorem 4.5 the quaternion algebra \mathcal{A} is not isomorphic to a matrix algebra. This implies by theorem 4.4 that $Stab(S_D)$ is a co-compact group. It also follows from construction that S_D is immersed and totally geodesic (and hence incompressible). \square

4.3 The Surface S_2

Hyperbolic upper half space \mathbb{H}^3 can be triangulated by tetrahedra (see for instance [T]). A totally geodesic surface in M_8 lifts to a plane in \mathbb{H}^3 . Suppose S_D lifts to a plane P_D , where $P_D \cap S_\infty^2$ is a round circle of euclidean radius \sqrt{D} , $D \in \mathbb{Z}_+$, centered at the origin. One can obtain a tessellation for P_D , by computing how P_D passes through the different tetrahedra in the triangulation of \mathbb{H}^3 . If P_D separates one vertex from the other three, we get a triangle. If P_D separates two vertices from the other two, we get a square.

The tessellation of P_2 is as given in Figure 2. If we also keep track of the labels on the sides of the tetrahedra, we can (by trial and error) find the identifications which close up the surface in M_8 . This gives us the fundamental domain for S_2 as shown in figure 3.

The identifications are :

$$\begin{aligned} g_1 &= \begin{pmatrix} 2\omega + 3 & -4 \\ 2 & 2\omega - 1 \end{pmatrix} = b^{-1}ab^2[a^{-1}, b]b^{-1}a^{-1} \\ g_2 &= \begin{pmatrix} 2\omega - 1 & 4\omega + 4 \\ 2\omega & 2\omega + 3 \end{pmatrix} = a^2b^{-1}a^{-2}b \\ g_3 &= \begin{pmatrix} -2\omega + 1 & 2\omega + 4 \\ -\omega + 1 & 2\omega + 3 \end{pmatrix} = ab^2ab^{-1} \\ g_4 &= \begin{pmatrix} 2\omega + 3 & 4\omega + 4 \\ 2\omega & 2\omega - 1 \end{pmatrix} = ba^{-2}b^{-1}a^2 \\ g_5 &= \begin{pmatrix} 2\omega + 3 & 4 \\ -2 & 2\omega - 1 \end{pmatrix} = ba^{-1}b^{-2}[a, b^{-1}]ba \\ g_6 &= \begin{pmatrix} 2\omega - 1 & 2\omega + 4 \\ -\omega + 1 & -2\omega - 3 \end{pmatrix} = a^{-1}b^{-2}a^{-1}b \end{aligned}$$

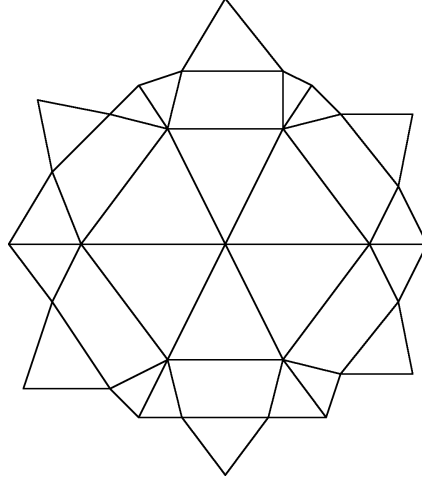


Figure 2: Tesselation of P_2

Where a and b are as before and $[a, b] = aba^{-1}b^{-1}$.

$\chi(S_2) = -2$, the surface is non-orientable (g_1, g_2, g_4 and g_5 are orientation reversing maps).

Hence S_2 is double covered by a genus 3 surface.

Remark : If g_i is an orientation reversing map, then g_i^2 lies in the Stabilizer subgroup. This means that g_i is of the form $\begin{pmatrix} \alpha & -D\beta \\ \bar{\beta} & -\bar{\alpha} \end{pmatrix}$ or $\begin{pmatrix} -\alpha & -D\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ (or its equivalent with respect to the projection $P : SL_2\mathbb{C} \rightarrow PSL_2\mathbb{C}$)

4.4 The Combinatorial Structure of the Lifts of S_2

Consider the surface S_2 . To find the elements (up to conjugacy) of the first fundamental group which cause self-intersections, we look at the fundamental domain of S_2 in \mathbb{H}^3 . This fundamental domain is made up of 6 squares and 24 triangles, each of which is contained in one of the tetrahedra which correspond to the fundamental domain of the figure-eight-knot complement. The fundamental domain of the figure-eight-knot complement is made up up 2 tetrahedra, T_1 and T_2 . The fundamental domain of S_2 is then contained in thirty tetrahedra, fifteen copies of T_1 and fifteen copies of T_2 . Hence elements of the fundamental group which cause self intersections will map one copy of a tetrahedra to another one of the fifteen copies of the same tetrahedra (we allow for the identity map). This allows us to construct a list of possible self intersections by checking the maps which send one copy of

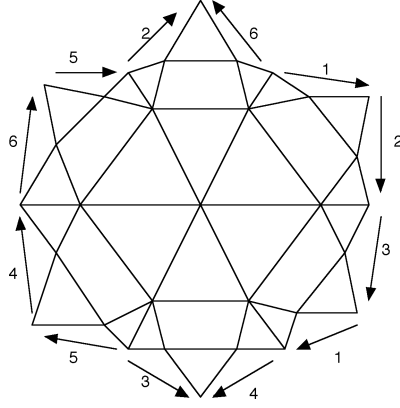


Figure 3: Fundamental domain of S_2

tetrahedron T_1 to another copy of the same tetrahedron. We can write $M_8 = T_1 \cup T_2$. The faces of T_1 have labels a_1, a_2, a_3 and a_4 , and we can label the edges as shown in figure 1. Similarly we label the faces of T_2 a'_1, a'_2, a'_3 and a'_4 . The labels assigned to the faces, have to be preserved by the isomorphism. Hence there is a unique isomorphism in the fundamental group of the figure-eight knot complement mapping one copy of the tetrahedron T_i to another copy of the same T_i . Fix the tetrahedron with vertices at $0, \omega, \omega + 1$ and ∞ . Construct maps $m[i]$ which map another copy of this tetrahedron back to this one. This gives us the list of maps as shown in table 1.

Table 1: List of $m[i]$, mapping one copy of T_1 to another.

$$\begin{aligned}
m[1] &= \{\{1, 0\}, \{0, 1\}\} \\
m[2] &= \{\{1, 1\}, \{0, 1\}\} \\
m[3] &= \{\{1, -1\}, \{0, 1\}\} \\
m[4] &= \{\{-\omega, 1\}, \{-\omega - 1, 1\}\} \\
m[5] &= \{\{0, \omega + 1\}, \{\omega, \omega + 2\}\} \\
m[6] &= \{\{0, \omega + 1\}, \{\omega, 2\}\} \\
m[7] &= \{\{1, 2\}, \{-\omega - 1, -2\omega - 1\}\} \\
m[8] &= \{\{1, 1\}, \{-\omega - 1, -\omega\}\} \\
m[9] &= \{\{1, 0\}, \{-\omega - 1, 1\}\} \\
m[10] &= \{\{1, -1\}, \{-\omega - 1, \omega + 2\}\} \\
m[11] &= \{\{-\omega, -1\}, \{\omega, \omega + 2\}\} \\
m[12] &= \{\{-\omega, -\omega + 1\}, \{-\omega - 1, -\omega\}\} \\
m[13] &= \{\{\omega, \omega - 1\}, \{\omega + 2, 2\omega + 2\}\} \\
m[14] &= \{\{-\omega, \omega + 1\}, \{-\omega - 1, \omega + 2\}\} \\
m[15] &= \{\{-\omega, \omega + 1\}, \{-\omega - 2, 2\}\}
\end{aligned}$$

Lemma 4.7. *All self intersections of the surface S_2 can be found by checking $m[i]^{-1}m[j]$.*

Proof: Suppose C is a double curve of S_2 . Then in M_8 one 2-simplex of the triangulation of S_2 intersects another. This implies that there has to be a covering translation $g \in \pi_1(M_8)$ so that $gP_2 \cap P_2 \neq \emptyset$ (where P_2 is the plane in the universal cover as defined before which projects down to S_2 .) But covering translations can be found by looking at two representatives of the same tetrahedron and using the labels on the tetrahedra to define a map. Note that once we know the labels on two copies of the same tetrahedron, we can pick three vertices and determine to which three uniquely determined vertices they have to be mapped. This uniquely defines a map in $\mathrm{PSL}_2\mathbb{C}$.

Any double curve comes from one simplex in the fundamental domain intersecting another simplex in the fundamental domain. Hence there are only finitely many combinations we need to check. Also note that one copy of T_i always has to be mapped to another copy of T_i , because the map has to lie in the fundamental group of the figure-eight-knot complement. Furthermore any map $T_1 \rightarrow gT_1, g \in \pi_1(M_8)$ which causes a self intersection of S_2 induces a map $T_2 \rightarrow gT_2, g \in \pi_1(M_8)$ which also causes a self intersection. To see this pass to the universal cover and consider the lift of the double curve, call it \tilde{C} . Note that there are two possible ways in which \tilde{C} meets the tessellation of P_2 . Either P_2 meets an edge of the tessellation of P_2 somewhere, or it has to pass from one simplex to another via vertices only. If the lifted double curve passes through an edge of the tessellation of P_2 , this corresponds to a double curve passing through a face of a tetrahedron, without loss of generality T_1 . Hence there is a corresponding double curve in the other tetrahedron T_2 . The other possibility is for the double curve to pass through two vertices of the tessellation of P_2 , and hence through two edges of the tetrahedron. But this double curve is the intersection of two different lifts of S_2 , which are both flat planes. This implies that the intersection must be a geodesic. This in turn implies that the double curve must coincide with an edge of the tessellation of P_2 . But this implies that the double curve also corresponds to a double curve in T_2 . Hence it suffices to check any possible map from one copy of T_1 to another copy of the same tetrahedron.

The $m[i]$ map a copy of one tetrahedron back to the reference tetrahedron. Hence all combinations $m[i]^{-1}m[j]$ will give a list of all maps causing double curves (up to multiplying by an element of the stabilizer subgroup). \square

Once we have this list of maps which correspond to double curves, we want to eliminate

duplication. Suppose $g \in \pi_1(M_8)$ is an element which maps one copy of the fundamental domain of S_2 in the universal cover onto another copy, hence causing a double curve in S_2 . Clearly g^{-1} corresponds to the same double curve. It is also clear that $g' = g\gamma$, where $\gamma \in \text{Stab}(S_2)$ corresponds to the same double curve. Finally, $g'' = \gamma g$, $\gamma \in \text{Stab}(S_2)$ corresponds to the same double curve. To see this, let P_2 be the lift of S_2 to the universal cover and let $p : \mathbb{H} \rightarrow M_8$ be the projection map. Consider $P_2 \cap \gamma g P_2$. $\gamma^{-1}(P_2 \cap \gamma g P_2) = \gamma^{-1} P_2 \cap g P_2 = C$, where $p(C)$ is the double curve. This implies that $P_2 \cap \gamma g P_2 = \gamma C$. But clearly $p(C) = p(\gamma C)$. We can use this information to reduce the list of 70 maps which cause double curves, to a list of 14 maps. Now it is easy to check the image of P_2 under these maps and deduce the height of the planes they define. By direct computation we find that the translates which cause self-intersections have radius $\sqrt{2}$, $\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{5}$. Hence we can put a maximal horoball at height $\sqrt{2}$, which will not intersect any of the translates of P_2 . This implies that in M_8 , the projection of the horoball at height $\sqrt{2}$ will be tangent to, but not intersecting, the surface S_2 . This gives us the following lemma.

Lemma 4.8. *Let M_8 be the figure-eight knot complement. Let $p : \widetilde{M}_8 \rightarrow M_8$ be the coveringmap from the universal cover \widetilde{M}_8 down to M_8 . Let $S_2 = p(P_2)$. For all $\epsilon > 0$, the projection of the horosphere at height $\sqrt{2} + \epsilon$ will be a boundary torus disjoint from the surface S_2 .*

If we let S_2^* denote the orientation double cover of S_2 , then clearly S_2^* also lifts to the plane P_2 in the universal cover. Furthermore any double curve of S_2^* double covers a double curve of S_2 . Hence we obtain the following corollary :

Corollary 4.9. *Let M_8 be the figure-eight knot complement. Let $p : \widetilde{M}_8 \rightarrow M_8$ be the coveringmap from the universal cover \widetilde{M}_8 down to M_8 . Let $S_2 = p(P_2)$. Let S_2^* denote the orientation double cover of S_2 . For all $\epsilon > 0$, the projection of the horosphere at height $\sqrt{2} + \epsilon$ will be a boundary torus disjoint from the surface S_2^* .*

4.5 Determining the height of the maximal horosphere

When doing surgery, we want to truncate the cusps so that we obtain a compact manifold with torus boundary components. To determine how far we can lower the horosphere in the universal cover without meeting the convex hull of our surface we can use combinatorial methods as described in the previous section, but there is a slightly easier method for totally geodesic surfaces.

Recall that the lift to the universal cover, which we called P_D , meets the sphere at infinity in a round circle centered at the origin of the deck in the upper half-space model.

There is a lemma in Maclachlan and Reid which gives us some general facts about primitive triples and discriminants :

Lemma 4.10. *Let $\mathcal{C}, \mathcal{C}'$ be represented by triples (a, B, c) and (a', B', c') respectively and let $T\mathcal{C} = \mathcal{C}'$ with $T \in \Gamma_d = PSL_2(\mathcal{O}_d)$ acting on the triple as a linear fractional transformation.*

1. (a, B, c) is primitive if and only if (a', B', c') is primitive.
2. If (a, B, c) is primitive, then $D(\mathcal{C}) = \mathcal{D}(\mathcal{C}')$

Proof: Define $\Sigma_d = \{\text{circles } \mathcal{C} \text{ represented by primitive triples in } \mathcal{O}_d\}$. Let \mathcal{C} be represented by the primitive triple (a, B, c) . Define $\Phi(a, B, c) = \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix}$.

Φ defines a bijection from Σ_d to

$$\mathcal{H}_d = \left\{ \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} \mid a, c \in \mathbb{Z}, B \in \mathcal{O}_d, ac - |B|^2 = -D, (a, B, c) \text{ is primitive} \right\}$$

$T \in \Gamma_d$ acts on \mathcal{H}_d by the following action :

$$\begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} \longrightarrow T \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} T^*, \text{ where } T^* \text{ is the complex conjugate transpose of } T$$

Proof of (1) : Suppose (a, B, c) is primitive (assume $b_1 \equiv b_2 \equiv 0 \pmod{2}$), the argument when $b_1 \equiv b_2 \equiv 1 \pmod{2}$ is similar). If T acts on $\Phi(a, B, c)$ and $\Phi(a', B', c')$ is not primitive, then we can write

$$T \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} T^* = \begin{pmatrix} a' & B' \\ \bar{B}' & c' \end{pmatrix} = p^2 \begin{pmatrix} a'' & B'' \\ \bar{B}'' & c'' \end{pmatrix} \text{ for some prime } p.$$

$$\text{But we know that } T^{-1} \begin{pmatrix} a' & B' \\ \bar{B}' & c' \end{pmatrix} T^{-1*} = p^2 T^{-1} \begin{pmatrix} a'' & B'' \\ \bar{B}'' & c'' \end{pmatrix} T^{-1*} = \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix}$$

This would imply that (a, B, c) was not primitive, which is a contradiction. Hence (a', B', c') must be primitive as well.

Proof of part (2) : This follows from the fact that $Det\Phi(a, B, c) = Det\Phi(a', B', c') = -D$.

□

Note that the hemisphere P_D has Euclidean height \sqrt{D} . Above lemma implies the following result.

Lemma 4.11. *Let P_D be a plane in \mathbb{H}^3 which meets the sphere at infinity in a circle of radius \sqrt{D} centered at the origin and projects down to a closed surface. Let $\Gamma_d = PSL_2(\mathcal{O}_d)$.*

Then any translate of P_D under an element of Γ_d is a plane meeting the sphere at infinity in a circle with a radius of at most \sqrt{D} .

Proof: P_D meets the sphere at infinity in a circle \mathcal{C} . The triple associated with this sphere is $(1, 0, -D)$. This triple is clearly primitive. Any $g \in \Gamma_d$ will map this triple to another primitive triple with discriminant D by the previous lemma. A direct computation shows that the radius of this new hemisphere is $\sqrt{\frac{D}{(a')^2}}$ if we assume that a' is not zero. This shows that the maximal height occurs when $a' = \pm 1$ and the maximal height attained is \sqrt{D} .

Now assume that $a' = 0$. Then after translation, the circle passes through infinity. There is a parabolic element which causes the original circle to intersect itself. Apply this parabolic to the translated circle. Then we must see a lift of a double curve with one end at infinity. Infinity is a fixed point, hence some power of this parabolic must map the lift of the double curve to itself. This implies that the fundamental group of the surface contains a parabolic. But closed surfaces do not contain parabolics. Hence $a' \neq 0$. \square

This lemma shows directly that

Lemma 4.12. *Let M be the figure-eight knot complement. Let P be the horoball neighborhood of the cusp chosen so that the boundary ∂P lifts to the horosphere with fixed point at infinity and at height $\sqrt{2} + \epsilon$ ($\epsilon > 0$). Then the boundary of P is disjoint from S_2 (and its orientation double cover S_2^*).*

5 Application I : the Figure-Eight Knot Complement

5.1 Incompressibility of S_2 after Surgery

The generators for the boundary torus are given by

$$y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } x = \begin{pmatrix} 1 & 4\omega \\ 0 & 1 \end{pmatrix}$$

where $\omega^3 = 1$ (see [T], chapter 4). It can be shown that the meridian and the longitude are given by :

$$\mu = y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \lambda = x + 2y = \begin{pmatrix} 1 & 4\omega + 2 \\ 0 & 1 \end{pmatrix}.$$

Hence the length of the lift of the surgery curve for (p, q) -surgery to the horosphere at height $\sqrt{2}$ is given by :

$$\frac{1}{\sqrt{2}}(p^2|\mu|^2 + q^2|\lambda|^2)^{\frac{1}{2}}$$

We can use the main theorem (theorem 3.7) to show that all but finitely many surgeries on the figure-eight-knot complement yield manifolds so that S_2^* is a π_1 -injective surface.

Remark Furthermore we can show that there is a metric in which the surface remains totally geodesic after most surgeries. But this does not allow us to conclude that the manifold is Virtually Haken. We only know this to be true when a totally geodesic surface is contained in a closed orientable hyperbolic manifold [L]. If we deform the metric to be hyperbolic, then we do not necessarily know what happens to the surface. Marden [M] showed that if the deformation is small enough, i.e. the surgery is large enough, then the surface will be quasi-Fuchsian. But in this specific case we do not know what would be considered a "large" surgery.

Note that due to the high degree of symmetry of the figure-eight knot it follows that $M_8(p/q) = M_8(-p/q)$.

Theorem 5.1. *The figure-eight knot contains a π_1 -injective surface, which will remain π_1 -injective after all but at most thirteen Dehn Surgeries. (These thirteen surgeries are : $\pm p/\pm q = 1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 1/2, 3/2$ or $5/2$)*

Proof: By lemma 4.11 we have that for all $\epsilon > 0$ the horoball of radius $\sqrt{2} + \epsilon$ which is disjoint from the surface P_2 . Hence we can remove this horoball, and using theorem 3.7 we can do surgery resulting in a manifold with negative curvature containing S_2 as an π_1 -injective surface if the surgery curve has length greater than 2π . For the figure-8-knot we need to know when $\frac{1}{\sqrt{2+\epsilon}}(p^2|\mu|^2 + q^2|\lambda|^2) > 2\pi$ In \mathbb{R}^2 , $\mu = (1, 0)$ and $\lambda = (0, 2\sqrt{3})$. Hence our length requirement becomes :

$$(p^2 + 12q^2)/(\sqrt{2} + \epsilon)^2 > 4\pi^2$$

Suppose that $\epsilon < \frac{1}{10}$ If $q = 1$, then $p^2 > (\sqrt{2} + \epsilon)^2 4\pi^2 - 12 (\approx 79)$ implies that we want $|p| > 8$.

If $q = 2$, then $p^2 > (\sqrt{2} + \epsilon)^2 4\pi^2 - 48 (\approx 43)$ implies that we want $|p| > 6$.

For $q \geq 3$, p can be any number.

This shows that all but above mentioned 13 surgeries on the figure-8-knot complement yield a manifold on which we can define a metric which is the old metric, except for on the horoball, where the metric is extended to a metric of negative curvature. With this metric, S_2 is still totally geodesic in the surgered manifold and by theorem 3.7 it is π_1 -injective. \square

5.2 The Exceptional Surgeries

Using the computer program GAP all but one of the exceptional surgeries can be dealt with. This results in the following theorem :

Theorem 5.2. *All surgeries on the figure-eight knot complement, except 1/0 surgery, give manifolds which contain a surface group.*

$M_8(8/1)$: Contains a subgroup of index 2 (Homology \mathbb{Z}_{20}), which contains a subgroup of index 5 with homology $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$.

$M_8(7/1)$: There is a subgroup of index 7 (homology $\mathbb{Z}_{14} \oplus \mathbb{Z}_{14}$), with a subgroup of index 7 containing $\mathbb{Z} \oplus \mathbb{Z}$.

$M_8(6/1)$: There is an index 10 subgroup with homology $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

$M_8(5/1)$: There are 2 subgroups of index 7 (both with homology \mathbb{Z}_{10}) and (at least) one of them has a subgroup of index 6 containing $\mathbb{Z} \oplus \mathbb{Z} \oplus G$, where G is a finite group.

$M_8(1/2)$: There is a unique subgroup of index 8 which has homology $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. This subgroup has a commutator subgroup with homology $\bigoplus_{i=1}^{16} \mathbb{Z} \oplus G$, where G is a torsion group.

$M_8(3/2)$: Has a unique subgroup of index 3. This subgroup has a subgroup of index 8 with homology $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$.

$M_8(5/2)$: There is a unique subgroup of index 5. This subgroup has a subgroup of index 7, with homology $\mathbb{Z} \oplus H$, where H is some non-trivial group.

If M has finite cover N with $\text{Rank } H_1(N; \mathbb{Z}) > 0$, then M is called **virtually \mathbb{Z} -representable** or **has virtually positive first Betti number**. With irreducibility this implies that M is virtually Haken. If M is Seifert Fibered, then M is virtually \mathbb{Z} -representable, for it has a finite cover which is a circle bundle over an orientable surface.

$M_8(0/1)$ is Solv manifold and is known to contain a surface group. $M_8(1/1)$, $M_8(2/1)$ and $M_8(3/1)$ are Small Seifert Fibered spaces, i.e. Seifert Fibered Spaces which fiber over the 2-sphere and have at most 3 exceptional fibers.

$M_8(4/1)$ is a Toroidal manifold.

This shows that the non-hyperbolic surgeries on the figure-eight-knot give virtually Haken manifolds.

We know that 1/0 surgery gives the 3-sphere and hence does not contain any surface groups.

6 Application II : the Whitehead Link

6.1 Surgery on Both Components of the Whitehead Link

Using the following diagram we can find the Wirtinger presentation of the Whitehead Link.

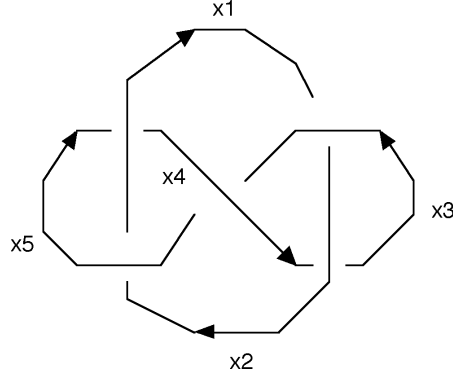


Figure 4: The Whitehead Link

The relations we get are : $x_1x_5 = x_4x_1, x_5x_1 = x_2x_5, x_3x_4 = x_4x_5, x_3x_1 = x_2x_3, x_2x_3 = x_4x_2$

These relations imply that we may rewrite x_2, x_3 and x_4 in terms of x_1 and x_5 . And we obtain a representation of the first fundamental group of the Whitehead Link (WL), which looks as follows :

$$\pi_1(WL) = \langle x_1, x_5 | x_1x_5x_1^{-1}x_5x_1x_5^{-1}x_1x_5 = x_5x_1x_5^{-1}x_1x_5x_1^{-1}x_5x_1 \rangle$$

Clearly both generators x_1 and x_5 are meridians of the boundary tori and hence are represented by parabolic elements when considering a representation in $PSL_2(\mathbb{C})$. Hence we may assume that the two generators are represented by matrices where

$$x_1 = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \text{ and } x_5 = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

The relation as given above implies that $p = 1$ and $t = 1 \pm i$.

Choose $t = 1 - i$. For one of the boundary tori we can read off that the meridian must be given by x_1 and the longitude by $x_5^{-1}x_1x_5x_1^{-1}x_5x_1x_5^{-1}x_1^{-1}$. Note that there is a homeomorphism which interchanges x_1 and x_5 . This implies that for the other boundary torus we get a meridian given by x_5 as expected and the longitude must be given by $x_1^{-1}x_5x_1x_5^{-1}x_1x_5x_1^{-1}x_5^{-1}$. By direct computation we see that the peripheral subgroups

must be given by :

$$\left\langle \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2-2i \\ 0 & 1 \end{pmatrix} \right) \right\rangle \text{ and (after conjugation) } \left\langle \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2-2i & 1 \end{pmatrix} \right) \right\rangle$$

Using results from Maclachlan and Reid [MR], we can show that there is a closed totally geodesic surface which lifts to a plane, P_3 , which traces out a circle of euclidean radius $\sqrt{3}$ in the universal cover \mathbb{H}^3 . We need the following lemma from [MR] :

Lemma 6.1. *Let $\mathcal{C} \in \Sigma_d = \{\text{circles } \mathcal{C} \text{ represented by primitive triples in } \mathcal{O}_d\}$. $D(\mathcal{C}) = D$. Then $Stab(\mathcal{C}, \Gamma_1)$ is non-co-compact if and only if D is not divisible by an odd power of a prime $e \equiv 3 \pmod{4}$*

Proof: Recall that $Stab(\mathcal{C}, \Gamma_1)$ is an arithmetic Fuchsian group, whose associated quaternion algebra is isomorphic to $(\frac{-1, D}{\mathbb{Q}})$, so that $Stab(\mathcal{C}, \Gamma_1)$ is non-co-compact if and only if $(\frac{-1, D}{\mathbb{Q}}) \cong M_2(\mathbb{Q})$. But $(\frac{-1, D}{\mathbb{Q}})$ has no finite ramification if and only if $D = n^2 D_0$, where D_0 is square free and for all prime p , $p|D_0$ implies $p \equiv 2$ or $p \equiv 1 \pmod{4}$. (A quaternion algebra \mathcal{A} is said to be ramified if when tensored with a field we obtain a division algebra, otherwise we say that \mathcal{A} is unramified.)

□

We can now show that P_3 maps down to a closed surface.

Corollary 6.2. *Let WL be the Whitehead link complement and let P_3 be a plane in \mathbb{H}^3 so that $P_3 \cap S_\infty^2$ is a circle of radius $\sqrt{3}$ centered at the origin in the upper half space model. P_3 maps down to a closed, orientable, π_1 -injective, immersed, totally geodesic surface, S_3 , in WL .*

Proof: $D = 3$ implies via the previous lemma that $Stab(\mathcal{C}_3, \Gamma_1)$ is co-compact. Hence P_3 will cover a closed surface in WL , call it S_3 , and we may assume it is orientable. It is totally geodesic by construction and hence π_1 -injective. □

Lemma 6.3. *Let WL be the Whitehead Link. S_3 as above. For all $\epsilon > 0$ the horosphere at height $\sqrt{3} + \epsilon$ maps down to a boundary torus disjoint from S_3 .*

Proof: This follows immediately from lemma 4.11. □

This implies the following theorem :

Theorem 6.4. *Let WL be the Whitehead link complement and let $WL(r_1, r_2)$ be the manifold obtained by performing $r_i = p_i/q_i$ Dehn surgery on the i -th torus boundary component ($i=1,2$).*

There exists a closed π_1 -injective surface in WL which remains π_1 -injective for all but at most 60 surgeries on each link component.

(no $\pm(p_i/q_i)$ ($i = 1, 2$) belongs to the following collection of surgery slopes :

*{1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 9/1, 10/1,
1/2, 3/2, 5/2, 7/2, 9/2, 1/3, 2/3, 4/3, 5/3, 7/3, 8/3,
1/4, 3/4, 5/4, 7/4, 1/5, 2/5, 3/5, 4/5}*)

Proof: We have two cusps in the Whitehead link complement. The meridian of one of the cusps is represented by the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which has infinity as its fixed point. Hence truncating the cusp corresponds to removing a horoball with its fixed point at infinity in the universal cover. The meridian of the other cusp is given by $\begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix}$ which has its fixed point at zero. Truncating this cusp corresponds to removing a horoball with fixed point at zero in the universal cover. If we think of excising the horoball as taking a horosphere with the same fixed point and lowering it as much as possible, it is clear that we have to compute how far we can lower both horospheres without meeting each other or the lifted surface P_3 . It is clear that P_3 is determined by $|z|^2 - 3 = 0$ which has $(1, 0, -3)$ as its primitive triple and $D = 3$. By lemma 4.11 we see that for all $\epsilon > 0$ the horoball at height $\sqrt{3} + \epsilon$ will be disjoint from P_3 and its translates. If we apply the map $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then we switch the roles of zero and infinity and we map the plane $|z|^2 - 3 = 0$ to $3|z|^2 - 1 = 0$. This new plane still has $D = 3$, so that any element of the fundamental group of the Whitehead link complement will move the surface away from a horosphere at height $\sqrt{3} + \epsilon$. This implies that the surgery conditions are the same for both cusps. Note that the representations of the peripheral groups tell us that we may choose the meridian to have length 1 and the longitude to have length 2. This gives $p_i^2 + 4q_i^2 > 3(2\pi)^2$ as the condition for surgery. (ϵ can be chosen as small as we want, hence this is the surgery condition to consider.) This gives the excluded surgeries as mentioned in the theorem. \square

6.2 Twist Knots

If we consider the manifolds obtained by Dehn surgery on just one component of the Whitehead link and denote the resulting manifold by $WL_{(p,q)}$, then a proposition from a

paper by Hodgson, Meyerhoff and Weeks [HMW] gives us the following information:

Proposition 6.5. *$WL(p/q)$ and $WL(p'/q')$ are homeomorphic if and only if $(p/q) = \pm(p'/q')$*

When we perform $1/n$ surgery on an unknotted components of a link, we obtain a knot or link complement. To see this note that if one of the link components is unknotted, then this implies that we can view $1/n$ surgery on that component as performing n meridinal twists on the complimentary solid torus followed by trivially filling in the unknotted component. This shows that we will obtain a knot complement in S^3 . In our setting, $1/n$ surgery on one of the components of the Whitehead link results in a twist knot. Given the correct choice of generators we obtain a $2n$ -twist knot after $1/n$ surgery when $n > 0$ and a $2n - 1$ -twist knot if $n < 0$. From work in [R] it follows that infinitely many twist knots can't contain closed, immersed, totally geodesic surfaces.

Theorem 6.6. *(Reid) There exist infinitely many twist knots which contain no closed totally geodesic surface and exactly one commensurability class of non-closed totally geodesic surface.*

The surgery theorem from the previous section immediately implies the following corollary :

Corollary 6.7. *All k -twist knots where $k > 10$ contain a closed, immersed, π_1 -injective surface.*

Proof: As before, there exists a closed, immersed, π_1 -injective totally geodesic surface in the Whitehead Link complement which lifts to the plane P_3 . Using the same methods as before, we remove a horoball neighborhood of the cusp which lifts to a horoball with fixed point at infinity and horosphere boundary at height $\sqrt{3}$. Note that due to the symmetry of the Whitehead Link we need not take into consideration which component we are filling. Up to homeomorphism we may assume we are filling the component which corresponds to the vertex at infinity. Restricting ourselves to $(1/n)$ surgery we obtain as our condition for surgery :

$$1 + 4n^2 > 3(2\pi)^2$$

Hence we need to exclude $|n| \leq 5$. But a $(1/n)$ surgery results in $2n$ extra crossings. Hence for a k -twist knots we need $k > 10$. □

In figure 5 we see the result of $1/n$ surgery on one component of the Whitehead link. It is fairly easy to see that this type of surgery does not change the surgery description on the remaining component. This also follows from Chapter 9H in Rolfsen [Ro]. This section analyses what happens when we twist about one unknotted component. Suppose we have a link given by $L = L_1 \cup \dots \cup L_n$, and suppose that one of the link components, say L_1 , is the unknot. The complement of an open neighborhood of L_1 is a solid torus. Assume L_1 bounds a disc D , so that the other components pass through in straight segments. We can construct a homeomorphism of the complement of the link by performing a meridinal twist on the solid torus. Each straight segment passing through D is then replaced by a helix twisting through a collared neighborhood of D . The new link $L' = L_1 \cup L'_2 \cup \dots \cup L'_n$ gives the same manifold after surgery if we revise the surgery coefficients in the appropriate manner.

Let $r_i = p_i/q_i$ be the surgery coefficient for the i -th link component. A proof is outlined in [Ro] which shows that the new surgery coefficients are as follows:

Proposition 6.8. *Let r_i be the surgery coefficient for the i -th link component. Let t be the number of right-handed twists about the unknotted component L_i ($t < 0$ for left-handed twists). Then the new surgery coefficients are given by :*

$$r'_i = (t + \frac{1}{r_i})^{-1} \text{ for the unknotted component of the twist.}$$

$$r'_j = r_j + t(\text{lk}(L_i, L_j))^2 \text{ for the other components.}$$

Sketch of Proof: Let D be the disk bounded by L_i , such that the other components pass through a collar of D in straight line segments. Suppose we perform t right handed twists about the unknotted component L_i . This implies that the straight line segments in the collar of D are replaced by a twisted helix. Suppose that the surgery coefficient on L_i was given by (p_i, q_i) before performing the twists. The twisting operation will take the meridian to the meridian, and it will take the longitude to one longitude plus t meridians. Hence the new surgery coefficient will be $(p_i, tp_i + q_i) = (1, t + \frac{q_i}{p_i})$.

For the other link components we see a slightly different picture. Let L_j be a link component passing through the disk D bounded by L_i . The surgery modification corresponds to performing a meridinal twist on the solid torus which is the complement of L_i . Let tu be the number of segments of L_j passing through D in an upward direction (assign one direction the label up, and the other direction the label down). Similarly let td be the number of segments of L_j passing through D in the downward direction. Note that every twist will contribute u upward segments and d downward segments. Assign the under-crossing a

± 1 according to the standard way of assigning a sign given a oriented link. Let λ_j be the longitude of L_j . We wish to compute the linking number of λ'_j with L'_j , where L'_j and λ'_j are the new link component and its longitude after modification. Every upward piece of λ'_j contributes $u - d$, while every downward bit of λ'_j contributes $d - u$. All other crossings correspond to crossings present when looking at the crossing of λ_j . But λ_j is a preferred longitude, hence those crossings cancel one another. Thus we see that

$$lk(\lambda'_j, L'_j) = tu(u - d) + td(d - u) = t(u - d)^2 = t(lk(L_i, L_j)^2).$$

Note that the twisting action sends the meridian of L_j to the meridian, and the longitude to a longitude plus t times $lk(L_i, L_j)^2$ meridians. This implies that the new surgery coefficient is given by $(p'_j, q'_j) = (p_j + tq_j(lk(L_i, L_j)^2), q_j) = (p_j, q_j) + t(lk(L_i, L_j)^2)$ \square

Hence if we are doing a $1/n$ surgery on the unknotted component, then we want to do $-n$ twists, so that the surgery description on this component becomes $1/0$ and we can just trivially fill in this torus boundary component. In the case of the Whitehead Link $lk(L_i, L_j) = 0$, hence the surgery description on the (altered) second component is not changed.

This implies that we can apply the arguments from theorem 6.4 out of the previous section again to show that even though this surface in the twist knots is not totally geodesic, it will remain π_1 -injective after all but 60 surgeries.

Corollary 6.9. *All k -twist knots where $k > 10$ contain a closed, immersed, π_1 -injective surface which remains π_1 -injective after all but at most sixty surgeries.*

($\pm(p/q) \neq \{1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 9/1, 10/1, 1/2, 3/2, 5/2, 7/2, 9/2, 1/3, 2/3, 4/3, 5/3, 7/3, 8/3, 1/4, 3/4, 5/4, 7/4, 1/5, 2/5, 3/5, 4/5\}$)

Proof: To see this, think of (p, q) surgery on the twist-knot as surgery on both components of the Whitehead Link, where one component has surgery coefficients (p, q) and the other has surgery coefficients $(1, n)$. By the previous theorem the resulting manifolds are homeomorphic. Hence the result follows immediately from Theorem 6.4. \square

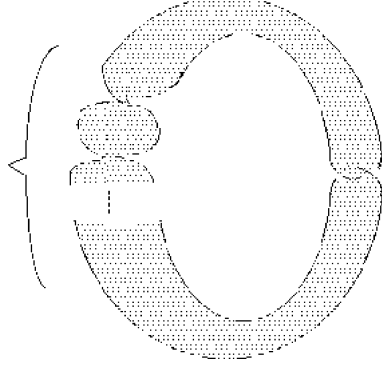


Figure 5: A twist knot

7 Application III : The Borromean Rings

7.1 Surgery on all Components of the Borromean Rings

It is easy to find the Wirtinger presentation of the fundamental group of the Borromean Rings. Denote the Borromean Rings complement by BR . Then we see that

$$\pi_1(BR) = \langle x_1, x_2, x_3 | x_3 x_2 x_3^{-1} x_1 x_3 x_1^{-1} = x_1 x_3 x_1^{-1} x_2, x_3 x_2 x_3^{-1} x_1 = x_2 x_1 x_2^{-1} x_3 x_2 x_3^{-1} \rangle$$

Hilden, Lozano and Montesinos [HLM] show that the representation into $PSL_2(\mathbb{Z}[i])$ is given by

$$x_1 \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, x_2 \rightarrow \begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix}, x_3 \rightarrow \begin{pmatrix} 1 & 0 \\ -2i & 1 \end{pmatrix}$$

It is easy to check that this representation satisfies the relations in the Wirtinger presentation. To find the euclidean length of the meridian and the longitude, we note first that for the three different boundary tori we can read off the peripheral subgroups. We get $\langle x_1, x_2^{-1} x_3 x_2 x_3^{-1} \rangle$, $\langle x_2, x_3^{-1} x_1 x_3 x_1^{-1} \rangle$, $\langle x_3, x_1^{-1} x_2 x_1 x_2^{-1} \rangle$. If we conjugate each meridian-longitude pair so that the fixed point lies at infinity, then in each case we obtain a representation where the meridian is given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the longitude by $\begin{pmatrix} 1 & -2i \\ 0 & 1 \end{pmatrix}$.

Hence the length of the meridian and longitude will be $\frac{1}{\sqrt{h}}$ for the meridian and $\frac{2}{\sqrt{h}}$ for the longitude, depending on the height h of the horosphere which covers the boundary

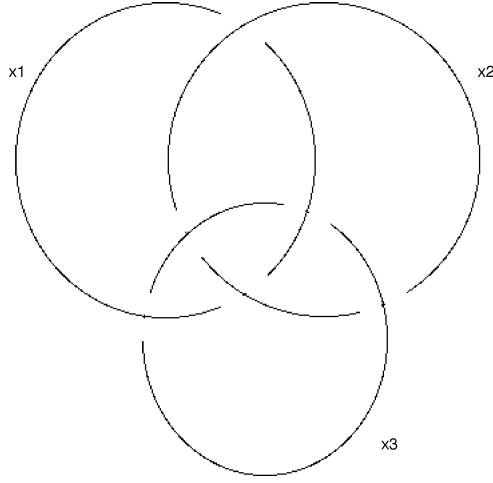


Figure 6: The Borromean Rings

torus in the truncated manifold we are performing surgery on.

We have the following lemma :

Lemma 7.1. *Let BR be the Borromean Rings complement and let P_3 be a plane in \mathbb{H}^3 so that $P_3 \cap S_\infty^2$ is a circle of radius $\sqrt{3}$ centered at the origin in the upper half space model. The plane P_3 in the universal cover maps down to a closed, orientable, totally geodesic surface in BR .*

Proof: Same argument as for the Whitehead Link complement. □

Theorem 7.2. *Let BR be the Borromean Rings complement and let $BR(r_1, r_2, r_3)$ be the manifold obtained by performing $r_i = (p_i, q_i)$ Dehn surgery on the i -th torus boundary component ($i=1,2,3$). $BR(r_1, r_2, r_3)$ contains a closed, immersed, π_1 -injective surface whenever no $(\pm p_i, \pm q_i)$ ($i = 1, 2, 3$) belongs to the following collection of surgery slopes :*

{1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 9/1, 10/1,
1/2, 3/2, 5/2, 7/2, 9/2, 1/3, 2/3, 4/3, 5/3, 7/3, 8/3,
1/4, 3/4, 5/4, 7/4, 1/5, 2/5, 3/5, 4/5}

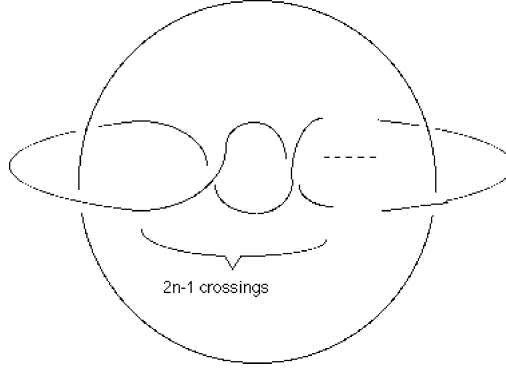


Figure 7: The link obtained by doing $1/n$ surgery on one of the components of the Borromean Rings.

7.2 Links and Knots obtained by filling one or two components of the Borromean Rings

If we denote the manifold obtained by not filling in a cusp at all by $M_{(\infty, \infty)}$. Then we can describe the manifold we obtain by filling in two cusps of the Borromean Rings by $BR_{(p_1, q_1), (p_2, q_2), (\infty, \infty)}$. Note that due to the symmetry of the Borromean Rings, there is no ambiguity. Similarly we can let $BR_{(p_1, q_1), (\infty, \infty), (\infty, \infty)}$ denote the manifold obtained by filling one cusp. If we restrict our attention to $(p_1, q_1) = (1, n)$ and $(p_2, q_2) = (1, m)$, then we know the manifold thus obtained does not have torsion and hence is a knot or link complement. We obtain the following results :

Corollary 7.3. *Let BR be the Borromean Rings complement and let $BR(1/n, \infty, \infty)$ be the link complement obtained by performing $1/n$ Dehn surgery on one of the torus boundary components. $BR(1/n, \infty, \infty)$ contains a closed, immersed, π_1 -injective surface whenever $(1/\pm n)$ does not belong to the following collection of surgery slopes :*

$$\{1/0, 1/1, 1/2, 1/3, 1/4, 1/5\}$$

Furthermore this surface will remain π_1 -injective after Dehn surgery on the remaining components whenever $(\pm p_i, \pm q_i)$ ($i = 1, 2$) do not belong to the following collection of surgery slopes :

$$\{1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 9/1, 10/1,$$

$$1/2, 3/2, 5/2, 7/2, 9/2, 1/3, 2/3, 4/3, 5/3, 7/3, 8/3,$$

$$1/4, 3/4, 5/4, 7/4, 1/5, 2/5, 3/5, 4/5\}$$

Proof: This follows immediately from the fact that the horoball neighborhoods can be chosen so that the lift of the boundary torus lies at height $\sqrt{3}$. By lemma 4.11, this horoball neighborhood will not intersect the closed surface we obtain by projecting down the plane P_3 . Hence by theorem 3.7 this closed, immersed, π_1 -injective surface remains π_1 -injective after surgery, unless $(1, n)$ belongs to the collection of short surgeries.

$lk(L_i, L_j) = 0$. Hence the surgery description on the components we are not filling remain unchanged. The theorem then easily follows from the fact that the surgery coefficients on the 2-link are the same as the surgery coefficients on the pre-images of the link components before performing the $(1/n)$ filling. \square

Corollary 7.4. *Let BR be the Borromean Rings complement and let $BR(1/n, 1/m, \infty)$ be the knot complement obtained by performing $1/n$ and $1/m$ Dehn surgery on two of the torus boundary components. $BR(1/n, 1/m, \infty)$ contains a closed, immersed, π_1 -injective surface whenever $(1/\pm n)$ and $(1/\pm m)$ do not belong to the following collection of surgery slopes :*

$$\{1/0, 1/1, 1/2, 1/3, 1/4, 1/5\}$$

Further more this surface will remain π_1 -injective after surgery on the remaining component whenever $(\pm p_i, \pm q_i)$ ($i = 1, 2$) do not belong to the following collection of surgery slopes :

$$\{1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 9/1, 10/1,$$

$$1/2, 3/2, 5/2, 7/2, 9/2, 1/3, 2/3, 4/3, 5/3, 7/3, 8/3,$$

$$1/4, 3/4, 5/4, 7/4, 1/5, 2/5, 3/5, 4/5\}$$

Proof: $lk(L_i, L_j) = 0$. The theorem now easily follows from the fact that the surgery coefficients on the knot is the same as the surgery coefficient on a link component in the

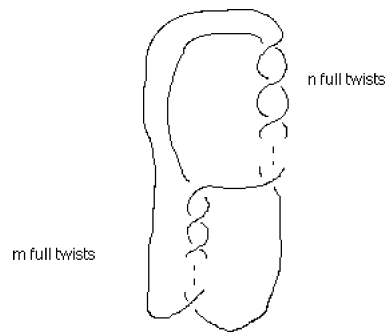


Figure 8: The knot obtained by performing $(1/n, 1/m, \infty)$ surgery on BR

pre-images before performing the $(1/n)$ and $(1/m)$ fillings on the remaining two link components. □

8 Surface Groups in Some Surgered Arithmetic Manifolds.

$PSL_2(\mathcal{O}_d)$ is called a Bianchi group. We know via the "Cuspidal Cohomology Problem" (see [B2]) that for any arithmetic manifold d belongs to a finite set of integers: $d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}$. Examples of knots and links for Bianchi Groups corresponding to $d = 1, 2, 3, 7, 11, 15$ and 23 can be found in [H],[MR],[NR] and [B2]. The method of proof used in section 4.2 to show the existence of closed totally geodesic surfaces can be extended to manifolds whose fundamental group is contained in one of the other Bianchi groups. To prove the case $d = 2$ we need a theorem from [O'M]. But first we need to review some algebra.

Suppose V is a vectorspace, then we say B is a **symmetric bilinear form** on V if $B : V \times V \rightarrow F$ is a map so that: $B(x, y + z) = B(x, y) + B(x, z)$, $B(ax, y) = aB(x, y)$ and $B(x, y) = B(y, x)$.

We define $Q(x) = B(x, x)$ to be a **quadratic map**.

Example 2. Suppose V is a quaternion algebra, then we can define $B(x, y) = \frac{1}{2}(x\bar{y} + y\bar{x})$. This then implies that $Q(x) = \frac{1}{2}(x\bar{x} + \bar{x}x) = x\bar{x} = n(x)$, where $n(x)$, the norm of x , is quadratic map.

V is said to have an **orthogonal splitting**, $V = V_1 \perp V_2 \perp \dots \perp V_r$, if $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$, where $B(V_i, v_j) = 0$.

The notation $V \cong \langle a_1 \rangle \perp \langle a_2 \rangle \perp \dots \perp \langle a_n \rangle$, $a_i \in F$ means that there is a basis x_1, x_2, \dots, x_n so that $Q(x_i) = a_i$

Example 3. Let $V = \left(\frac{\alpha, \beta}{\mathbb{Q}}\right)$, then we can write

$V = \{a_0 + a_1i + a_2j + a_3ij | a_i \in \mathbb{Q}, i^2 = \alpha, j^2 = \beta \text{ and } ij = -ji\}$. Using $Q(x) = N(x)$ gives $V \cong \langle 1 \rangle \perp \langle -\alpha \rangle \perp \langle -\beta \rangle \perp \langle \alpha\beta \rangle$. It is easy to see that the basis is $1, i, j, ij$, and clearly $N(1) = 1, N(i) = i(-i) = -\alpha, N(j) = j(-j) = -\beta, N(ij) = ij(-ij) = i^2j^2 = \alpha\beta$.

Note that $\langle \alpha \rangle \perp \langle \beta \rangle$ represents 1 means that there are $x, y \in F$ so that $\alpha x^2 + \beta y^2 = 1$. And when C is a quaternion algebra, then C^0 is the quadratic space of pure quaternions. Where an element x is said to be pure if and only if $\bar{x} = -x$. The theorem we need is the following :

Theorem 8.1. Let α and β be non-zero elements of a field F . Then the following assertions are equivalent:

1. $\left(\frac{\alpha, \beta}{F}\right)$ is algebra isomorphic to $\left(\frac{1, -1}{F}\right)$.
2. $\left(\frac{\alpha, \beta}{F}\right)$ is not a division algebra.
3. $\left(\frac{\alpha, \beta}{F}\right)$ is isotropic.
4. $\left(\frac{\alpha, \beta}{F}\right)^0$ is isotropic.
5. $\langle \alpha \rangle \perp \langle \beta \rangle$ represents 1.
6. $\alpha \in N_{E/F}E$, where $E = F(\sqrt{\beta})$.

Sketch of Proof: (see [O'M]) (1 \Rightarrow 2) : Let $1, i, j, ij$ be the basis for the quaternion algebra $\left(\frac{1, -1}{F}\right)$. Then $(i + j)^2 = 0$, hence $\left(\frac{1, -1}{F}\right)$ is not a division algebra, and thus $\left(\frac{\alpha, \beta}{F}\right)$ is not a division algebra.

(2 \Rightarrow 3) : If $\left(\frac{\alpha, \beta}{F}\right)$ is not a division algebra then there is at least one $x \neq 0$ which does not have an inverse. Then $N(x) = 0$ (for if $N(x) \neq 0$ then $((N(x)^{-1}\bar{x})x = N(x)^{-1}N(\bar{x}) = N(x)^{-1}N(x) = 1$ and $x((N(x)^{-1}\bar{x}) = 1$, so that the inverse of x exists.) But this shows that $\left(\frac{\alpha, \beta}{F}\right)$ is isotropic.

(3 \Rightarrow 4) : It can be shown that if U is a regular ternary subspace of a regular quaternary space V , and V has discriminant 1, then V is isotropic if and only if U is isotropic.

(4 \Rightarrow 5) : We are given that $\langle -\alpha \rangle \perp \langle -\beta \rangle \perp \langle \alpha\beta \rangle$ is isotropic. Hence $\langle \alpha^2\beta \rangle \perp \langle \alpha\beta^2 \rangle \perp \langle -\alpha^2\beta^2 \rangle$ is isotropic. So that $\langle \beta \rangle \perp \langle \alpha \rangle \perp \langle -1 \rangle$ is isotropic. This implies that $\langle \beta \rangle \perp \langle \alpha \rangle$ represents 1.

(5 \Rightarrow 6) : $\langle \alpha \rangle \perp \langle \beta \rangle$ represents 1 implies that $\langle \alpha \rangle \perp \langle \beta \rangle \perp \langle -1 \rangle$ is isotropic. Hence $\langle -\alpha \rangle \perp \langle -\beta \rangle \perp \langle 1 \rangle$ is isotropic. Hence $\langle 1 \rangle \perp \langle -\beta \rangle$ represents α . So there exist $\gamma, \delta \in F$ such that $\alpha = \gamma^2 - \beta\delta^2$. Then in $E = F(\sqrt{\beta})$ we have $\alpha = (\gamma + \delta\sqrt{\beta})(\gamma - \delta\sqrt{\beta}) = N_{E/F}(\gamma + \delta\sqrt{\beta}) \in N_{E/F}E$

(6 \Rightarrow 1) : We have $\gamma, \delta \in F$ such that $\alpha = N_{E/F}(\gamma + \delta\sqrt{\beta})$. So $\alpha = \gamma^2 - \beta\delta^2$, hence $\langle -\alpha \rangle \perp \langle -\beta \rangle \perp \langle 1 \rangle$ is isotropic. Hence $\langle -\alpha \rangle \perp \langle -\beta \rangle \perp \langle \alpha\beta \rangle$ is isotropic. So there is an x in $\left(\frac{\alpha, \beta}{F}\right)^0$ with $x \neq 0$ and $N(x) = 0$. Hence x is not invertible. So $\left(\frac{\alpha, \beta}{F}\right)$ can not be a division algebra. Hence by Wedderburn's theorem $\left(\frac{\alpha, \beta}{F}\right)$ is algebra isomorphic to $M_2(F)$ and hence to $\left(\frac{1, -1}{F}\right)$. \square

This allows us to prove all the cases of the following theorem :

Theorem 8.2. *Suppose M is an arithmetic 3-manifold (hyperbolic) and $\pi_1(M)$ is a subgroup of $PSL_2(\mathcal{O}_d)$, then the plane of radius \sqrt{D} will project down to a closed, totally*

geodesic surface for the following values :

If $d = 1$ and $D \equiv 3 \pmod{4}$.

If $d = 2$ and $D \equiv 5 \pmod{25}$.

If $d = 3$ and $D \equiv 2 \pmod{3}$.

If $d = 5$ and $D \equiv 2 \pmod{5}$.

If $d = 6$ and $D \equiv 2 \pmod{3}$.

If $d = 7$ and $D \equiv 3 \pmod{7}$ or $D \equiv 5 \pmod{7}$.

If $d = 11$ and $D \equiv 2 \pmod{11}$.

If $d = 15$ and $D \equiv 2 \pmod{3}$.

If $d = 19$ and $D \equiv 2 \pmod{19}$.

If $d = 23$ and $D \equiv 5 \pmod{23}$.

If $d = 31$ and $D \equiv 3 \pmod{31}$.

If $d = 39$ and $D \equiv 2 \pmod{3}$.

If $d = 47$ and $D \equiv 5 \pmod{47}$.

If $d = 71$ and $D \equiv 7 \pmod{71}$.

Proof: $d = 1$: This follows from lemma 6.1 by Maclachlan and Reid.

$d = 2$: D is a square free number. Hence the equation $|\alpha|^2 - D|\beta|^2 = 1$ has infinitely many integral solutions. (This is Pell's Equation.) This implies that $Stab(S_D)$ is non-elementary. From [M,MR] it then follows that $Stab(S_D)$ is finite covolume. $Stab(S_D)$ is a maximal subgroup of $PSL_2(\mathcal{O}_2)$ and $\pi_1(M)$ is a subgroup of finite index, hence the fundamental group of the surface will be of finite index in $Stab(S_D)$. Hence to show that the surface is compact, it suffices to show that $Stab(S_D)$ is cocompact. We want to show that when $D \equiv 5 \pmod{25}$, then $\left(\frac{-2, D}{\mathbb{Q}}\right)$ is a division algebra. By theorem 8.1 this is equivalent to showing that when $D \equiv 5 \pmod{25}$, $\langle -2 \rangle \perp \langle D \rangle$ does not represent 1. This means we have to show that there are no $x, y \in \mathbb{Q}$ so that $-2x^2 + Dy^2 = 1$. We can rewrite this condition as : there are no $x, y \in \mathbb{Z}$ so that $-2x^2 + Dy^2 = n^2$. When $D \equiv 5 \pmod{25}$, we see that we must have that $-2x^2 + 5y^2 \equiv n^2 \pmod{25}$. But it can be shown that $-2x^2 \equiv 0, \pm 2, \pm 8, \pm 12, \pm 18$ or $\pm 22 \pmod{25}$ and that $5x^2 \equiv 0$ or $\pm 5 \pmod{25}$. But $n^2 \equiv 0, \pm 1, \pm 4, \pm 6, \pm 9$ or $\pm 11 \pmod{25}$. Hence $-2x^2 + 5y^2 \equiv n^2 \pmod{25}$ implies that x^2, y^2 and n^2 must all be congruent to 0 modulo 25. But we may assume the original equation to have been in reduced form. Hence this is a contradiction.

Thus $\left(\frac{-2, D}{\mathbb{Q}}\right)$ is a division algebra, and hence not a matrix algebra. Let \mathcal{C} denote the circle traced out by P_D at the sphere at infinity. Then the above argument implies via

theorem 4.4 that $Stab(\mathcal{C}, PSL_2(\mathcal{O}_2))$ is cocompact. Hence P_D maps down to a compact surface.

$d = 3$: See proof of 4.6.

Similar proofs hold for $d = 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71$

□

To compute which surgery curves have sufficiently large length, we need to know the length of the longitude. If we normalize the length of the meridian, we can find a lower bound on the length of the longitude.

Theorem 8.3. *Let M be an arithmetic knot or link complement so that $\pi_1(M) \leq PSL_2(\mathcal{O}_d)$. Let λ be a longitude. If we have normalized so that the length of the meridian associated to λ is equal to 1, then the length of the longitude $L(\lambda) = \frac{\sqrt{d+1}}{2}$ if $d \equiv 3 \pmod{4}$ and $L(\lambda) = \sqrt{d+1}$ otherwise.*

Proof: We can conjugate the representation of the fundamental group, so that the meridian is given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The longitude then has to be given by $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. We know that α has to be a complex number, otherwise the meridian and the longitude can't generate the peripheral subgroup. Furthermore it has to be an algebraic integer because it is an element of $PSL_2(\mathcal{O}_d)$. Hence the length L is given by the magnitude of α .

When $d \equiv 3 \pmod{4}$ then $\alpha = \frac{1}{2}(a + b\sqrt{-d})$, where $a, b \in \mathbb{Z}$. Hence $L = \frac{1}{2}\sqrt{a^2 + db^2}$ and the lower bound for L is $\frac{\sqrt{d+1}}{2}$.

Otherwise $\alpha = (a + b\sqrt{-d})$, where $a, b \in \mathbb{Z}$. So that $L = \sqrt{a^2 + db^2}$ and the lower bound for L is $\sqrt{d+1}$. □

This gives us the following theorem :

Theorem 8.4. *Suppose M is an arithmetic 3-manifold (hyperbolic) and $\pi_1(M)$ is a subgroup of $PSL_2(\mathcal{O}_d)$. Let L be the minimal length of the longitude after normalizing the meridian to have length 1. Let D be chosen so that the surface covered by the plane of a given radius \sqrt{D} is a closed totally geodesic surface. This surface will remain π_1 -injective if for the p/q surgeries on each component the surgery condition holds as shown in the table. We also give an upperbound on the number of bad surgeries (where the surface might compress).*

d	D	L	<i>surgery condition</i>	<i># bad surgeries</i>
1	3	$\sqrt{2}$	$p^2 + 2q^2 > 3(2\pi)^2$	83
2	5	$\sqrt{3}$	$p^2 + 3q^2 > 5(2\pi)^2$	111
3	2	1	$p^2 + q^2 > 2(2\pi)^2$	77
5	2	$\sqrt{6}$	$p^2 + 6q^2 > 2(2\pi)^2$	33
6	2	$\sqrt{7}$	$p^2 + 7q^2 > 2(2\pi)^2$	31
7	3	$\sqrt{2}$	$p^2 + 2q^2 > 3(2\pi)^2$	83
11	2	$\sqrt{3}$	$p^2 + 3q^2 > 2(2\pi)^2$	45
15	2	2	$p^2 + 4q^2 > 2(2\pi)^2$	39
19	2	$\sqrt{5}$	$p^2 + 5q^2 > 2(2\pi)^2$	35
23	5	$\sqrt{6}$	$p^2 + 6q^2 > 5(2\pi)^2$	79
31	3	$\sqrt{8}$	$p^2 + 8q^2 > 3(2\pi)^2$	41
39	2	$\sqrt{10}$	$p^2 + 10q^2 > 2(2\pi)^2$	25
47	5	$\sqrt{12}$	$p^2 + 12q^2 > 5(2\pi)^2$	55
71	7	$\sqrt{18}$	$p^2 + 18q^2 > 7(2\pi)^2$	63

Ian Agol pointed out to me that (at least for orientable manifolds) 2π can be replaced by 6. This would show that we obtain the following upperbounds for number of bad surgeries for each link component : 75 (d=1), 101 (d=2), 69 (d=3), 31 (d=5), 29 (d=6), 75 (d=7), 39 (d=11), 37 (d=15), 35 (d=19), 75 (d=23), 37 (d=31), 23 (d=39), 51 (d=47), 59 (d=71).

Theorem 8.5. *Let L be an arithmetic link whose fundamental group is contained in a Bianchi group . If L' is a knot or link complement obtained via $1/n_i$ surgery on one or more unknotted components of L and none of the $1/n_i$ surgeries are bad, then L' will contain a closed π_1 -injective surface. Furthermore this surface which will remain π_1 -injective after all but a finite number of surgeries on each component of L' . The number of bad surgeries is determined by the Bianchi group of the original arithmetic link complement.*

Proof: If $lk(L_i, L_j) = 0$, then the other surgery description remain the same. To determine the bad surgeries, we would just find the bad surgeries for the component in the original link.

If $lk(L_i, L_j) \neq 0$, then the surgery descriptions are modified by $r'_j = r_j + t(lk(L_i, L_j))^2$.

Hence the number of bad surgeries will not change. \square

9 Appendix

9.1 The fundamental domain of the surface S_5

One can obtain a tessellation for P_D , by computing how P_D passes through the different tetrahedra in the triangulation of \mathbb{H}^3 . If P_D separates one vertex from the other three, we get a triangle. If P_D separates two vertices from the other two, we get a square.

The tessellation of \widetilde{S}_5 is as given in Figure 6.

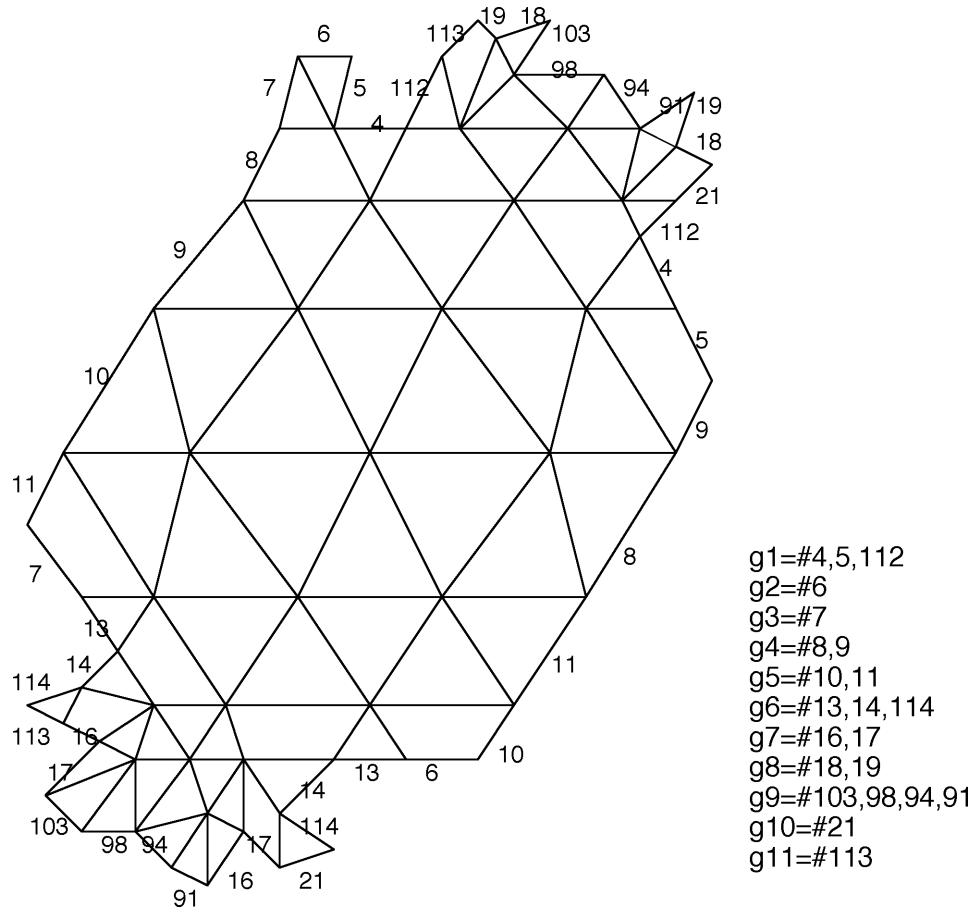


Figure 9: Fundamental domain of S_5

g_1 is the isometry glueing sides 4, 5 and 112.

g_2 is the isometry glueing side 6.

g_3 is the isometry glueing side 7.

g_4 is the isometry glueing sides 8 and 9.

g_5 is the isometry glueing sides 10 and 11.

g_6 is the isometry glueing sides 13, 14 and 114.

g_7 is the isometry glueing sides 16 and 17.

g_8 is the isometry glueing sides 18 and 19.

g_9 is the isometry glueing sides 103, 98, 94 and 91.

g_{10} is the isometry glueing side 21.

g_{11} is the isometry glueing side 113.

The identifications are :

$$\begin{aligned}
g_1 &= \begin{pmatrix} 4\omega & -5\omega + 5 \\ \omega + 2 & -4\omega - 4 \end{pmatrix} = [b^{-1}, a][b, a^{-1}]b^{-1}a^{-1}b^{-1}[a, b^{-1}] \\
g_2 &= \begin{pmatrix} 6 & -15\omega - 5 \\ 3\omega + 2\omega & 6 \end{pmatrix} = [b, a^{-1}][a, b]ba^2b^{-1}a[b^{-1}, a^{-1}] \\
g_3 &= \begin{pmatrix} -4\omega + 1 & -10\omega - 10 \\ 2\omega & 4\omega + 5 \end{pmatrix} = [b, a^{-1}][b^{-1}, a^{-1}]b^{-1}a^{-1}b[a^{-1}, b]a[b^{-1}, a^{-1}] \\
g_4 &= \begin{pmatrix} 2\omega & 5\omega + 5 \\ \omega & 2\omega + 2 \end{pmatrix} = [b, a^{-1}][b^{-1}, a^{-1}]b^{-1}a^4[b^{-1}, a^{-1}] \\
g_5 &= \begin{pmatrix} 2\omega + 2 & 5\omega + 5 \\ \omega & 2\omega \end{pmatrix} = [a, b]a^4b^{-1}[a, b][a, b^{-1}] \\
g_6 &= \begin{pmatrix} 4\omega + 4 & 5\omega - 5 \\ -\omega - 2 & -4\omega \end{pmatrix} = [b, a^{-1}]b^{-1}a^{-1}b^{-1}[a, b^{-1}][a^{-1}, b] \\
g_7 &= \begin{pmatrix} -2\omega - 12 & -25\omega - 25 \\ -5\omega & -2\omega + 10 \end{pmatrix} = [b, a^{-1}][a^{-1}, b^{-1}]ba^{-1}ba[a, b^{-1}][a^{-1}, b]a \\
g_8 &= \begin{pmatrix} -2\omega + 10 & -25\omega - 25 \\ -5\omega & -2\omega - 12 \end{pmatrix} = a[b^{-1}, a][b, a^{-1}]aba^{-1}b[b, a][a, b^{-1}] \\
g_9 &= \begin{pmatrix} 9 & 20\omega + 20 \\ -4\omega - 2 & 9 \end{pmatrix} = [b^{-1}, a]ba^{-2}[b, a^{-1}]b^{-1}a[b^{-1}, a^{-1}]b[a^{-1}, b] \\
g_{10} &= \begin{pmatrix} -4\omega - 15 & 30\omega + 30 \\ -6\omega & 4\omega - 11 \end{pmatrix} = [b, a^{-1}][b^{-1}, a]a^3b^{-1}a^{-1}b^{-1}[a, b^{-1}] \\
g_{11} &= \begin{pmatrix} 4\omega - 11 & 30\omega + 30 \\ -6\omega & -4\omega - 15 \end{pmatrix} = [b, a^{-1}]b^{-1}a^{-1}b^{-1}a^3[a^{-1}, b][a, b^{-1}]
\end{aligned}$$

Where a and b are as before and $[a, b] = aba^{-1}b^{-1}$.

$\chi(S_2) = -4$, the surface is non-orientable (g_7, g_8, g_4 and g_5 are orientation reversing maps).

Hence S_2 is double covered by a genus 5 surface.

Remark : If g_i is an orientation reversing map, then g_i^2 lies in the Stabilizer subgroup. This means that g_i is of the form $\begin{pmatrix} \alpha & -D\beta \\ \beta & -\bar{\alpha} \end{pmatrix}$ or $\begin{pmatrix} -\alpha & -D\beta \\ \beta & \bar{\alpha} \end{pmatrix}$ (or its equivalent with respect to the projection $P : SL_2\mathbb{C} \rightarrow PSL_2\mathbb{C}$)

9.2 Double curves of S_2

We found the covering translations which cause self-intersections by the following method. We pick the tetrahedron with vertices at $0, 1, \omega + 1$ and ∞ and call it T . This represents one of the two tetrahedra in the fundamental polygon of the figure-8-knot complement. If we now look at the fundamental domain of S_2 , we see 15 copies of this tetrahedron. We construct 15 isometries, called $m[i]$, which map T to any one of the 15 copies (this includes the identity).

$$m[1] = \{\{1, 0\}, \{0, 1\}\}$$

$$m[2] = \{\{1, 1\}, \{0, 1\}\}$$

$$m[3] = \{\{1, -1\}, \{0, 1\}\}$$

$$m[4] = \{\{-\omega, 1\}, \{-\omega - 1, 1\}\}$$

$$m[5] = \{\{0, \omega + 1\}, \{\omega, \omega + 2\}\}$$

$$m[6] = \{\{0, \omega + 1\}, \{\omega, 2\}\}$$

$$m[7] = \{\{1, 2\}, \{-\omega - 1, -2\omega - 1\}\}$$

$$m[8] = \{\{1, 1\}, \{-\omega - 1, -\omega\}\}$$

$$m[9] = \{\{1, 0\}, \{-\omega - 1, 1\}\}$$

$$m[10] = \{\{1, -1\}, \{-\omega - 1, \omega + 2\}\}$$

$$m[11] = \{\{-\omega, -1\}, \{\omega, \omega + 2\}\}$$

$$m[12] = \{\{-\omega, -\omega + 1\}, \{-\omega - 1, -\omega\}\}$$

$$m[13] = \{\{\omega, \omega - 1\}, \{\omega + 2, 2\omega + 2\}\}$$

$$m[14] = \{\{-\omega, \omega + 1\}, \{-\omega - 1, \omega + 2\}\}$$

$$m[15] = \{\{-\omega, \omega + 1\}, \{-\omega - 2, 2\}\}$$

If we now look at $m[i]^{-1}.m[j]$, $1 \leq i, j \leq 15$, we see that this corresponds to all possible intersections of the surface S_2 with itself. The following program in Mathematica will compute all possible combinations $m[i]^{-1}.m[j]$ and check that the isometry causes a self-intersection.

```
(* enter a list of the entries in the matrix, All w.r.t the basis {1,i} *)
new[z_] := Module[ {a={z[[7]],z[[8]]}, b=-{z[[3]],z[[4]]}, c=-{z[[5]],z[[6]]}, d={z[[1]],z[[2]]},
move = {-(a[[1]]2 + a[[2]]2)(x2 + y2) + 2(a[[1]]b[[1]] + a[[2]]b[[2]])x +
2(a[[1]]b[[2]] - a[[2]]b[[1]])y + (b[[1]]2 + b[[2]]2)
-2((c[[1]]2 + c[[2]]2)(x2 + y2) + 2(c[[1]]d[[1]] + c[[2]]d[[2]])x
+2(c[[1]]d[[2]] - c[[2]]d[[1]])y + (d[[1]]2 + d[[2]]2))};
p = Solve[{move[[1]] == 0, x2 + y2 - 2 == 0}, {x, y}/N];
```

$s = \{\mathbf{x}, \mathbf{y}\} / .\mathbf{p};$

```

For[k = 1, k < 16, k ++, For[i = k + 1, i < 16, i ++,
n=Expand[Inverse[m[i]].m[k]];
l=new[{Re[n[[1]][[1]]],Im[n[[1]][[1]]], Re[n[[1]][[2]]],Im[n[[1]][[2]]],
Re[n[[2]][[1]]],Im[n[[2]][[1]]], Re[n[[2]][[2]]],Im[n[[2]][[2]]] }];
If[Im[s[[1]][[1]]]==0 && Im[s[[1]][[2]]]==0,
AppendTo[listA,n];
listA=Union[listA;,;]]
InputForm[listA]

```

```

For[k = 1, k < 71, k ++,
g[k] = {{2 * (Im[listA[[k]][[1]][[1]]]/Sqrt[3]) * v + ((Im[listA[[k]][[1]][[1]]]/Sqrt[3]) +
Re[listA[[k]][[1]][[1]]]), 2 * (Im[listA[[k]][[1]][[2]]]/Sqrt[3]) * v +
((Im[listA[[k]][[1]][[2]]]/Sqrt[3]) + Re[listA[[k]][[1]][[2]]])},
{2 * (Im[listA[[k]][[2]][[1]]]/Sqrt[3]) * v + ((Im[listA[[k]][[2]][[1]]]/Sqrt[3]) +
Re[listA[[k]][[2]][[1]]]), 2 * (Im[listA[[k]][[2]][[2]]]/Sqrt[3]) * v +
((Im[listA[[k]][[2]][[2]]]/Sqrt[3]) + Re[listA[[k]][[2]][[2]]])}};
Print["g", k, "] = ", g[k];]

```

(The output is on the next page)

From these isometries we can compute the radius of the circle at infinity this translate traces out. We get the following list:

```

-1 ± x + x2 ± Sqrt[3]y + y2 = 0, (D = 2)
-1 ± 2x + x2 + y2 = 0, (D = 2)
1 + x2 + 2Sqrt[3]y + y2 = 0, (D = 2)
1 ± 2x + 2x2 - 2Sqrt[3]y + 2y2 = 0, (D = 1/2)
1 ± 3x + x2 ± Sqrt[3]y + y2 = 0, (D = 2)
1 ± 4x + 2x2 + 2y2 = 0, (D = 1/2)
2 ± 2x + x2 ± 2Sqrt[3]y + y2 = 0, (D = 2)
2 - 4x + x2 + y2 = 0, (D = 2)
5 + 2x2 - 4Sqrt[3]y + 2y2 = 0, (D = 1/2)
5 - x + x2 - 3Sqrt[3]y + y2 = 0, (D = 2)
5 + x + x2 ± 3Sqrt[3]y + y2 = 0, (D = 2)

```

$$5 \pm 4x + x^2 + 2Sqrt[3]y + y^2 = 0, (D = 2)$$

$$5 + 5x + x^2 + Sqrt[3]y + y^2 = 0, (D = 2)$$

$$5 - 5x + x^2 \pm Sqrt[3]y + y^2 = 0, (D = 2)$$

$$5 \pm 6x + 2x^2 + 2Sqrt[3]y + 2y^2 = 0, (D = 1/2)$$

$$7 \pm x + 5x^2 - 7Sqrt[3]y + 5y^2 = 0, (D = 2/25)$$

$$7 - 4x + 2x^2 - 4Sqrt[3]y + 2y^2 = 0, (D = 1/2)$$

$$7 \pm 4x + 2x^2 + 4Sqrt[3]y + 2y^2 = 0, (D = 1/2)$$

$$7 - 8x + 2x^2 + 2y^2 = 0, (D = 1/2)$$

$$7 \pm 10x + 5x^2 + 4Sqrt[3]y + 5y^2 = 0, (D = 2/25)$$

$$7 \pm 11x + 5x^2 + 3Sqrt[3]y + 5y^2 = 0, (D = 2/25)$$

Hence all the translates which cause self-intersections of S_2 are hemispheres of Euclidean diameter $\sqrt{2}$, $1/\sqrt{2}$ or $\sqrt{2}/5$.

$$\begin{aligned}
g[1] &= \{-1, -2\}, \{0, -1\} \\
g[2] &= \{0, -1 - \omega\}, \{-\omega, 2 + \omega\} \\
g[3] &= \{0, 1 + \omega\}, \{\omega, 2 + \omega\} \\
g[4] &= \{0, 1 + \omega\}, \{\omega, 2 + 2\omega\} \\
g[5] &= \{1, -2\}, \{1 + \omega, -1 - 2\omega\} \\
g[6] &= \{1, -1\}, \{0, 1\} \\
g[7] &= \{1, -1\}, \{1 + \omega, -\omega\} \\
g[8] &= \{1, 0\}, \{-1 - \omega, 1\} \\
g[9] &= \{1, 0\}, \{1 + \omega, 1\} \\
g[10] &= \{1, 1\}, \{0, 1\} \\
g[11] &= \{1, 1\}, \{-1 - \omega, -\omega\} \\
g[12] &= \{1, 1\}, \{1 + \omega, 2 + \omega\} \\
g[13] &= \{1, 2\}, \{0, 1\} \\
g[14] &= \{1, 2\}, \{-1 - \omega, -1 - 2\omega\} \\
g[15] &= \{1, 3\}, \{-1 - \omega, -2 - 3\omega\} \\
g[16] &= \{2, -3 - \omega\}, \{-\omega, \omega\} \\
g[17] &= \{2, -3 - \omega\}, \{2 + \omega, -2 - 2\omega\} \\
g[18] &= \{2, -1 - \omega\}, \{-\omega, 0\} \\
g[19] &= \{2, -1 - \omega\}, \{2 + \omega, -\omega\} \\
g[20] &= \{2, 1 - \omega\}, \{-\omega, -\omega\} \\
g[21] &= \{2, 1 - \omega\}, \{2 + \omega, 2\} \\
g[22] &= \{2, -3 + \omega\}, \{\omega, -2\omega\} \\
g[23] &= \{2, -1 + \omega\}, \{\omega, -\omega\} \\
g[24] &= \{2, 1 + \omega\}, \{\omega, 0\} \\
g[25] &= \{2, 3 + \omega\}, \{\omega, \omega\} \\
g[26] &= \{-1 - 2\omega, -2\}, \{1 + \omega, 1\} \\
g[27] &= \{-1 - 2\omega, -3 - 2\omega\}, \{1 + \omega, 2 + \omega\} \\
g[28] &= \{-1 - 2\omega, -1 + 2\omega\}, \{1 + \omega, -\omega\} \\
g[29] &= \{1 + 2\omega, 2\}, \{-1 - \omega, -1\} \\
g[30] &= \{1 + 2\omega, 3 + 2\omega\}, \{-1 - \omega, -2 - \omega\} \\
g[31] &= \{-2 - \omega, -3 - 2\omega\}, \{-\omega, -\omega\} \\
g[32] &= \{-\omega, -1\}, \{\omega, 2 + \omega\} \\
g[33] &= \{-\omega, -1\}, \{1 + \omega, 1\} \\
g[34] &= \{-\omega, 1\}, \{-\omega, 2 + \omega\} \\
g[35] &= \{-\omega, 3\}, \{\omega, -2 + \omega\} \\
g[36] &= \{-\omega, -1 - \omega\}, \{1 + \omega, 2 + \omega\} \\
g[37] &= \{-\omega, 1 - \omega\}, \{-1 - \omega, -\omega\} \\
g[38] &= \{-\omega, 1 - \omega\}, \{-\omega, 2\} \\
g[39] &= \{-\omega, -1 + \omega\}, \{1 + \omega, -\omega\} \\
g[40] &= \{-\omega, 3 + \omega\}, \{\omega, -2\} \\
g[41] &= \{-\omega, -1 + 2\omega\}, \{1 + \omega, -1 - 2\omega\} \\
g[42] &= \{2 - \omega, -3 - 3\omega\}, \{2 + \omega, -2\omega\} \\
g[43] &= \{\omega, 1\}, \{-1 - \omega, -1\} \\
g[44] &= \{\omega, 1 - \omega\}, \{-1 - \omega, \omega\} \\
g[45] &= \{\omega, 1 + \omega\}, \{-1 - \omega, -2 - \omega\} \\
g[46] &= \{\omega, -3 - 2\omega\}, \{-\omega, 2 + \omega\} \\
g[47] &= \{\omega, 1 - 2\omega\}, \{-1 - \omega, 1 + 2\omega\} \\
g[48] &= \{\omega, 3 + 2\omega\}, \{\omega, 2 + \omega\} \\
g[49] &= \{\omega, 3 + 3\omega\}, \{\omega, 2 + 2\omega\} \\
g[50] &= \{\omega, 3 + 3\omega\}, \{1 + \omega, 2 - \omega\} \\
g[51] &= \{2 + \omega, 1\}, \{-\omega, -\omega\} \\
g[52] &= \{2 + \omega, 1\}, \{1 + \omega, 1\} \\
g[53] &= \{2 + \omega, -1 - \omega\}, \{-\omega, 0\} \\
g[54] &= \{2 + \omega, -1 - \omega\}, \{1 + \omega, -\omega\} \\
g[55] &= \{2 + \omega, 3 + \omega\}, \{-\omega, -2\omega\} \\
g[56] &= \{2 + \omega, 3 + \omega\}, \{1 + \omega, 2 + \omega\} \\
g[57] &= \{2 + \omega, -3 - 2\omega\}, \{-\omega, \omega\} \\
g[58] &= \{2 + \omega, -3 - 2\omega\}, \{1 + \omega, -1 - 2\omega\} \\
g[59] &= \{-2\omega, -3 - \omega\}, \{\omega, 2 + \omega\} \\
g[60] &= \{-2\omega, 3 + \omega\}, \{-2 - \omega, -2\omega\} \\
g[61] &= \{-2\omega, 3 + \omega\}, \{-\omega, 2 + \omega\} \\
g[62] &= \{-2\omega, -3 - 3\omega\}, \{\omega, 2 + 2\omega\} \\
g[63] &= \{1 - 2\omega, -3 - 2\omega\}, \{2, -1 - 2\omega\} \\
g[64] &= \{2 + 2\omega, 1 + \omega\}, \{\omega, 0\} \\
g[65] &= \{2 + 2\omega, 3 + \omega\}, \{-2 - \omega, -2\} \\
g[66] &= \{2 + 2\omega, -3 - 3\omega\}, \{\omega, -2\omega\} \\
g[67] &= \{2 + 2\omega, -1 - 3\omega\}, \{-2 - \omega, 2 + 2\omega\} \\
g[68] &= \{2 + 2\omega, 3 + 3\omega\}, \{\omega, \omega\}
\end{aligned}$$

$$g[69] = \{\{3 + 2\omega, 1 - 2\omega\}, \{-2, 1 + 2\omega\}\}$$

$$g[70] = \{\{-2 - 3\omega, -3 + \omega\}, \{2 + \omega, -2\omega\}\}$$

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