

Immersed and Virtually Embedded Boundary slopes in Arithmetic Manifolds.

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Abstract

Given a Bianchi Group $\Gamma_d = PSL_2(\mathcal{O}_d)$, and a Hyperbolic manifold M , where $\pi_1(M)$ is of finite index in Γ_d , we show that all boundary slopes are realized as the boundary slope of an immersed totally geodesic surface and hence are virtually embedded boundary slopes.

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1 Introduction

A *slope* on a torus T is the isotopy class of an essential un-oriented simple closed curve, γ , on T . We say that a boundary slope is an *embedded boundary slope* if there is an embedded, compact, orientable, π_1 -injective surface so that the boundary of this surface consists of loops in T parallel to γ . It is known that a knot only has finitely many embedded boundary slopes [H,82]. A boundary slope is an *immersed boundary slope* if there exists a properly immersed, compact, orientable, π_1 -injective surface whose boundary consists of loops parallel to γ . We also assume that the surface cannot be homotoped into the boundary by a proper homotopy. If this immersion is covered by an embedding in some finite cover, then we say that the slope is a *virtually embedded boundary slope*. Baker and Cooper [BC,97] have shown that for the figure eight knot, all slopes with even numerator are virtually embedded boundary slopes. Oertel [O,95] and Maher [Ma,99] have shown that there are manifolds where every slope is an immersed boundary slope.

In this paper we will focus on Bianchi groups, and manifolds whose fundamental group is a finite index subgroup of a Bianchi group. Let d be a square free positive integer, and let \mathcal{O}_d denote the ring of integers in the quadratic number field $\mathbb{Q}(\sqrt{-d})$. The groups $\Gamma_d = PSL_2(\mathcal{O}_d)$ are called Bianchi groups. We let $\omega = \sqrt{-d}$ when $d \not\equiv 3 \pmod{4}$, and $\omega = (1 + \sqrt{-d})/2$ when $d \equiv 3 \pmod{4}$. Consider the lattice in the complex plane generated by 1 and ω . Consider the cusp at infinity. A vertical plane passing through at least two lattice points in the complex plane will project down to a cusped totally geodesic surface. We say that any plane parallel to the plane passing through 0 and $n + m\omega$ has rational slope $\frac{m}{n}$. If we consider for instance those planes that pass through the origin and any other lattice point, we will obtain all possible rational boundary slopes. If the classnumber is greater than or equal to 2, we can adapt our argument to show that all slopes at any given boundary component is a virtually embedded boundary slope.

Work by Agol, Long and Reid [ALR,01] implies that this surface will lift to an immersed surface in a finite cover. Hence the boundary slopes are virtually embedded boundary slopes.

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Theorem 1.1. *Let M be a complete hyperbolic 3-manifold, so that $\pi_1(M)$ is a finite index subgroup of Γ_d for some value of d . If γ is a slope on a boundary torus of M , then γ is a virtually embedded boundary slope.*

Note that using Freedman tubing (see [Wu,02] for background), we can always construct a π_1 -injective, geometrically finite surface that meets only one of the cusps of the manifold.

Well known manifolds that belong to this category are the figure eight knot complement and the Whitehead link. Both come from a Bianchi group with class number 1, hence the arguments are fairly straight forward. For the figure eight knot complement we obtain the following corollary.

Corollary 1.2. *All rational slopes are virtually embedded boundary slopes in the figure eight knot complement.*

The Whitehead link has two boundary components, and there are different questions we can pose. We obtain the following corollaries. In the first result we consider surfaces with slopes on both boundary components of the link, while in the second result we consider surfaces meeting only one boundary component of the link.

Lemma 1.3. *Given a pair of immersed boundary slopes $(\frac{p}{q}, \frac{-p}{q})$ (one slope for each of the boundary components of the Whitehead link), there is a cusped totally geodesic surface with those boundary slopes. Further more this boundary slope pair is a virtually embedded boundary slope pair.*

Corollary 1.4. *Given an immersed boundary slopes $\frac{p}{q}$ on one of the boundary components of the Whitehead link, there is an immersed π_1 -injective surface with this boundary slopes (and no boundary slope on the other boundary component).*

It has long been known that fundamental groups of knot and link complements can only occur as finite index subgroups of finitely many Bianchi groups. There are several papers showing examples of such knot and link complements. (See for instance [B,01] and [H,83].) We have included examples where the Bianchi groups have class number 2 and 3 respectively. Examples of links in these Bianchi groups were given by Baker in [B,01] (a 2-component link, call it L_{15} , whose fundamental group is a finite index subgroup of Γ_{15} , and a 6-component link, call it L_{23} , whose fundamental group is a finite index subgroup of Γ_{23} . For the link L_{15} we obtain results similar to the results for the Whitehead link.

Lemma 1.5. *Given a pair of immersed boundary slopes $(\frac{p}{q}, \frac{-p}{q})$ for L_{15} (one slope for each of the boundary components of the link), there is a cusped totally geodesic surface with those boundary slopes. Further more this boundary slope pair is a virtually embedded boundary slope pair.*

Corollary 1.6. *Given an immersed boundary slopes $\frac{p}{q}$ on one of the boundary components of the link L_{15} , there is an immersed π_1 -injective surface with this boundary slopes (and no boundary slope on the other boundary component).*

For class number greater than or equal to 3 there are similar results. We can always find surfaces that have boundary on two of the cusp classes. It is not clear that there are necessarily totally geodesic surfaces meeting all cusps. In the L_{23} example we would need to identify what the cusp class of each of the link components is. Assuming we have this information we obtain the following result.

Lemma 1.7. *Suppose we're given a boundary slope $\frac{p}{q}$ on a boundary component of the link L_{23} corresponding to the infinity cusp, and a boundary slope $\frac{-p}{q}$ on a boundary component corresponding*

to one of the finite cusps. Then there is an immersed totally geodesic surface with these boundary slopes. Further more these boundary slopes are a virtually embedded boundary slopes.

Corollary 1.8. *Given an immersed boundary slopes $\frac{p}{q}$ on one of the boundary components of the link L_{23} , there is an immersed π_1 -injective surface with this boundary slopes (and no boundary slope on the other boundary components).*

The outline of the paper is as follows: In section 2.1 we give background and general results for the existence of totally geodesic surfaces in Bianchi groups. In section 2.1 we prove Theorem ??.

Section 3 discusses the figure eight knot example and we prove corollary ??.

Section 4 discusses the Whitehead link and we prove the results concerning boundary slopes for the Whitehead link.

In section 5 we look at the examples for Bianchi groups with larger class numbers, and we prove the results for the specific examples L_{15} and L_{23} .

2 Totally Geodesic Surfaces

2.1 Background and General Results

Suppose that M is an orientable, hyperbolic 3-manifold. Then $\pi_1(M)$ has a representation as a discrete and faithful, torsion-free subgroup of $PSL_2(\mathbb{C})$. An immersed π_1 -injective surface $g : S \rightarrow M$ determines a subgroup $G_*(\pi_1(S))$ of $\pi_1(M)$, and hence of $PSL_2(\mathbb{C})$. If this subgroup is conjugate to $PSL_2(\mathbb{R})$ then S is said to be *totally geodesic* (sometimes called *Fuchsian*). This is equivalent to saying that the limit set of $\pi_1(S)$ is a geometric circle.

Totally geodesic surfaces in Bianchi groups have been described in detail in papers by Maclachlan [M,86], and Maclachlan and Reid [MR,91]. We will give a short summary of the results that will be used in this paper.

Suppose that the limit set of $\pi_1(S)$ is a geometric circle \mathcal{C} . We have the following results:

Theorem 2.1. *(Maclachlan)*

The stabilizer subgroup

$$Stab(\mathcal{C}, \Gamma_d) = \{\gamma \in \Gamma_d \mid \gamma\mathcal{C} = \mathcal{C} \text{ and } \gamma \text{ preserves the components of } \mathbb{C} - \mathcal{C}\}$$

is a maximal arithmetic subgroup of Γ_d

Theorem 2.2. *(Maclachlan)*

Every maximal non-elementary Fuchsian subgroup of Γ_d is an arithmetic Fuchsian group arising from some quaternion algebra $\frac{-d, D}{\mathbb{Q}}$. In particular, they have finite co-volume.

Maclachlan goes on to show that all \mathcal{C} are of the form

$$a|z|^2 + Bz + \bar{B}\bar{z} + c = 0$$

Let H be a finitely generated subgroup of a group G . G is called H -subgroup separable if given any $g \in G \setminus H$, there exists a subgroup $K < G$ of finite index with $H < K$ and $g \notin K$. We have the following result from [ALR,01]:

Theorem 2.3. *(Agol, Long, Reid)*

All Bianchi groups are H -subgroup separable for geometrically finite subgroups.

This theorem implies that totally geodesic surfaces lift to embedded surfaces in some finite cover. Clearly this property also holds for any manifold whose fundamental group is a finite index subgroup of a Bianchi group.

Totally geodesic surfaces can be closed (see [Ba,01], [Ba,02] for instance) or have boundary. Bianchi groups act on hyperbolic space, and the cusps of the orbifold correspond to the number of ideal classes of the ring \mathcal{O}_d . This number is called the class number. If the class number is one, then all cusps are equivalent under a Γ_d action to the cusp at ∞ (See Swan [S,71] for example). If the class number is greater than one, we can find representatives for the cusps. We always choose ∞ as one of the cusps, and we call the other cusps *finite cusps*. We usually denote these by some $\frac{\alpha}{b}$, where $\alpha \in \mathcal{O}_d$ and $b \in \mathbb{Z}$. There are a finite number of these finite cusps.

Given $\lambda = \frac{\alpha}{b}$. The element $L = \begin{pmatrix} \alpha & 1 \\ b & 0 \end{pmatrix} \in GL_2(\mathbb{Q}(\sqrt{-d}))$ maps infinity to λ . $L^{-1}H(\lambda)$ is invariant under the group $L^{-1}Stab(\lambda)L$. The following proposition from [Vo,85] determines a fundamental domain for this groups of translations.

Proposition 2.4. (Vogtmann) *Let $\lambda = \frac{\alpha}{b}$ be a finite cusp. Then the group $L^{-1}Stab(\lambda)L$ acts on $\partial\mathbb{H}^3 \setminus \infty$ via translations by elements of the fractional ideal $b < \alpha, b >^{-2}$*

We will use this information to show that all rational slopes at finite cusps are virtually embedded slopes.

2.2 Slopes of Totally Geodesic Surfaces

For the purpose of this paper we are interested in cusped totally geodesic surfaces.

Theorem 2.5. *Let M be a complete hyperbolic 3-manifold, so that $\pi_1(M)$ is a finite index subgroup of Γ_d for some value of d . If γ is a slope on a boundary torus of M , then γ is a virtually embedded boundary slope.*

Proof: Case 1: The cusp(s) corresponding to infinity.
Suppose that $d \not\equiv 3 \pmod{4}$. Then the equation

$$Bz + \bar{B}\bar{z} = (nd + m\sqrt{-d})z + (nd - m\sqrt{-d})\bar{z} = 0$$

defines a plane that passes through 0 and $n + m\sqrt{d}$. This plane meets the cusp at infinity in a curve of slope m/n . Clearly we can construct a plane meeting the cusp at infinity in a curve of any rational slope m/n .

Similarly suppose that $d \equiv 3 \pmod{4}$. Then the equation

$$Bz + \bar{B}\bar{z} = (m\frac{d-1}{2} - n + (2n+m)\omega)z + (m\frac{d-1}{2} - n + (2n+m)\bar{\omega})\bar{z} = 0$$

defines a plane that passes through 0 and $n + m\omega$. This plane meets the cusp at infinity in a curve of slope m/n . Clearly we can construct a plane meeting the cusp at infinity in a curve of any rational slope m/n .

These surfaces has discriminant $D = |B|^2$. Where we assume that the triple (a, B, c) defining the surface is a primitive triple. (I.e. in this special case we assume that given $B = b_1 + b_2\sqrt{-d}$, when $d \not\equiv 3 \pmod{4}$ and $B = \frac{b_1}{2} + \frac{b_2}{2}\sqrt{-d}$, when $d \equiv 3 \pmod{4}$ that $gcd(b_1, b_2) = 1$). If (a, B, c) is not primitive, then we can simplify the equation for \mathcal{C} without changing the slope.

Note that the stabilizers of these surfaces contain the group generated by $\begin{pmatrix} 1 & (n+m\omega)t \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ (n-m\omega)t & 1 \end{pmatrix}$. Hence these groups are non-elementary. It now follows from theorems 2.1 and 2.2 that these surfaces have finite co-volume. For, $Stab(\mathcal{C})$ is a maximal subgroup of Γ_d , and $\pi_1(M)$ is of finite index in Γ_d . Hence the fundamental group of the totally geodesic surface will be of finite index in $Stab(\mathcal{C})$. Hence the surface is of finite co-volume.

It follows from theorem 2.3 that these surfaces lift to an embedded surface in a finite cover. This does not change the slope of the surface as we defined it.

Case 2: Finite cusps.

When the class number is strictly greater than 1, we have to consider finite cusps. Suppose a boundary torus lifts to a horosphere with fixed point at $\frac{\alpha}{b}$. It follows from results by Vogtman that we can conjugate the representation of the fundamental group by $L = \begin{pmatrix} \alpha & 1 \\ b & 0 \end{pmatrix}$ so that infinity and $\frac{\alpha}{b}$ are exchanged. The stabilizer of the cusp is generated as a lattice by $b < \alpha, b >^{-2}$. An easy computation shows that planes through any two points of this lattice will project down to a cusped totally geodesic surface. We will define the slope at this finite cusp to be $\frac{m}{n}$ if the plane is parallel to the plane passing through 0 and $n + m\omega$. A straight forward computation shows that a plane through $\frac{\alpha}{b}$ with slope $\frac{m}{n}$ at the infinity cusp corresponds to a plane with slope $\frac{-m}{n}$ at the finite cusp. This is a non-elementary subgroup, and an argument similar to the one used above also shows that this surface is finite co-volume and lifts to an embedding in some finite cover. \square

3 Application: The Figure Eight Knot Complement

Let M be the figure eight knot complement. $\pi_1(M) = \langle a, b | b^{-1}aba^{-1}b = ab^{-1}aba^{-1} \rangle$, where we may choose $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ \omega + 1 & 1 \end{pmatrix}$, where $\omega^3 = 1, \omega \neq 1$. (See [T,78] and [R,75] for instance.) In general, a boundary slope for a knot or link complement is given in terms of the meridian and the longitude. It can be shown that the meridian and the longitude for the figure eight knot complement are given by $\mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\lambda = \begin{pmatrix} 1 & 4\omega + 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2\sqrt{-3} \\ 0 & 1 \end{pmatrix}$.

Given a boundary slope $p\mu + q\lambda$, choose a geodesic representative. Then this slope lifts to a straight line in a horosphere. We may assume that the line passes through 0 and $p + 2q\sqrt{-3}$. There is a vertical plane passing through these two points and we obtain, after projection down to the manifold of this plane, our totally geodesic surface with the required boundary slope. Totally geodesic surfaces in the figure eight knot complement lift to an embedded surface in some finite cover (See [ALR,01]) This argument proves the following corollary.

Corollary 3.1. *All rational slopes are virtually embedded boundary slopes in the figure eight knot complement.*

4 Application: The Whitehead Link Complement

Using the Wirtinger presentation we can show that the fundamental group of the Whitehead Link is given by:

$$\pi_1(WL) = \langle a, b | aba^{-1}bab^{-1}ab = bab^{-1}aba^{-1}ba \rangle$$

Where we may choose $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix}$.

a is the meridian of one boundary torus, and b is the meridian of the other. Note that a has ∞ as a fixed point, and b has 0 as a fixed point. This means that ∞ and 0 are representatives of the respective cusps. There is no element of the fundamental group conjugating ∞ to 0 or vice versa. There are many elements of $PSL_2(\mathbb{Z}[i])$ that do however. Any element mapping ∞ to 0 must be of the form $\gamma = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$, where $b, c, d \in \mathbb{Z}[i]$ and $bc = -1$.

Consider any plane passing through the origin and some point in the lattice generated by 1 and i . This is a plane that maps down to a surface with boundary at both cusps. Such a plane is given by the equation $Bz + \bar{B}\bar{z} = 0$, where $B \in \mathbb{Z}[i]$. Applying γ^{-1} gives a plane given by the equation $B\bar{z} + \bar{B}z = b\bar{d}B + \bar{b}d\bar{B}$. This shows that if the slope at one cusp is $\frac{p}{q}$, then the slope at the other cusp is $\frac{-p}{q}$. This gives us the following corollary:

Lemma 4.1. *Given a pair of immersed boundary slopes $(\frac{p}{q}, \frac{-p}{q})$ (one slope for each of the boundary components of the Whitehead link), there is a cusped totally geodesic surface with those boundary slopes. Further more this boundary slope pair is a virtually embedded boundary slope pair.*

Proof: This follows from the main theorem. (Theorem ??) Consider all the planes passing through ∞ , 0 , and any other lattice point. These planes all project down to finite co-volume cusped surfaces. These surfaces meet both boundary components of the link by construction. A straight forward computation shows they have slope $(\frac{p}{q}, \frac{-p}{q})$. These surfaces are totally geodesic and hence lift to an embedding in some finite cover. \square

Remark: Note that $(\frac{0}{q}, \frac{0}{-q}) = (\frac{0}{q}, \frac{0}{q})$ and $(\frac{p}{0}, \frac{-p}{0}) = (\frac{p}{0}, \frac{p}{0})$, for these correspond to planes that meet the complex plane in the real and imaginary axis respectively.

Using results by Wu [W,02], we can perform a Freedman tubing at one of the cusps.

If the wrapping number is large enough, this surface will still be π_1 -injective and it has the property that its boundary only meets one of the cusps of the manifold. We can also perform the tubing at the appropriate boundaries in the finite cover in which the surface embeds. The projection of this surface will still be π_1 -injective, but this construction shows that this single boundary slope may not be virtually embedded.

Corollary 4.2. *Given an immersed boundary slopes $\frac{p}{q}$ on one of the boundary components of the Whitehead link, there is an immersed π_1 -injective surface with this boundary slopes (and no boundary slope on the other boundary component).*

Proof: Use the surfaces from corollary ???. Perform a Freedman tubing where the tubes are sufficiently long. \square

5 Examples where the classnumber is greater than 1.

It has long been known that fundamental groups of knot and link complements can only occur as finite index subgroups of finitely many Bianchi groups. There are several papers showing examples of such knot and link complements. (See for instance [B,01] and [H,83].) Baker gives an example (in [B,01]) of a 2-component link, call it L_{15} , whose fundamental group is a finite index subgroup of Γ_{15} . This Bianchi group has class number 2. We obtain results similar to the results for the Whitehead link.

Lemma 5.1. *Given a pair of immersed boundary slopes $(\frac{p}{q}, \frac{-p}{q})$ for L_{15} (one slope for each of the boundary components of the link), there is a cusped totally geodesic surface with those boundary slopes. Further more this boundary slope pair is a virtually embedded boundary slope pair.*

Proof: Γ_{15} has class number 2, and the ideal classes can be represented by ∞ and $\frac{\omega+1}{2}$. It follows from computations similar to those used to prove theorem that planes through $\infty, \frac{\omega+1}{2}$, parallel to lines through the origin and lattice points, will project down to finite covolume totally geodesic surfaces. These surfaces will meet both boundary components of the link by construction. A straight forward computation shows that we obtain slope pairs $(\frac{p}{q}, \frac{-p}{q})$. \square

Corollary 5.2. *Given an immersed boundary slopes $\frac{p}{q}$ on one of the boundary components of the link L_{15} , there is an immersed π_1 -injective surface with this boundary slopes (and no boundary slope on the other boundary component).*

Proof: This results from Freedman tubing the appropriate boundary components of the surfaces generated in the proof of lemma ?? \square

The situation becomes more complicated when the class number is greater than or equal to 3. Baker gives an example of a 6-component link (in [B,01]), call it L_{23} , whose fundamental group is a finite index subgroup of Γ_{23} . Note that Γ_{23} has class number 3, and the cusps may be represented by $\infty, \frac{\omega}{2}, \frac{\bar{\omega}}{2}$. In this example we would need to identify what the cusp class of each of the link components is.

Lemma 5.3. *Suppose we're given a boundary slope $\frac{p}{q}$ on a boundary component of the link L_{23} corresponding to the infinity cusp, and a boundary slope $\frac{-p}{q}$ on a boundary component corresponding to one of the finite cusps. Then there is an immersed totally geodesic surface with these boundary slopes. Further more these boundary slopes are a virtually embedded boundary slopes.*

Proof: Γ_{23} has class number 3, and the ideal classes can be represented by $\infty, \frac{\omega}{2}$ and $\frac{\bar{\omega}}{2}$. It follows from computations similar to those used to prove theorem that planes through $\infty, \frac{\omega}{2}$ (resp. $\frac{\bar{\omega}}{2}$), parallel to lines through the origin and lattice points, will project down to finite covolume totally geodesic surfaces. These surfaces will meet the boundary components of the link as specified in the hypotheses. A straight forward computation shows that we obtain slope pairs $(\frac{p}{q}, \frac{-p}{q})$. \square

Corollary 5.4. *Given an immersed boundary slopes $\frac{p}{q}$ on one of the boundary components of the link L_{23} , there is an immersed π_1 -injective surface with this boundary slopes (and no boundary slope on the other boundary component).*

Proof: This results from Freedman tubing the appropriate boundary components of the surfaces generated in the proof of lemma ?? □

Remark: There exists a totally geodesic surface with boundary slope $\frac{n}{0}$ on each boundary of the link L_{23} . Where n is the smallest integer that results in a closed curve.

6 Further Questions

For the Whitehead link it is possible to find one surface that realizes the boundary slope pair $(\frac{p}{q}, \frac{-p}{q})$. This leads to the obvious question: is there a π_1 -injective surface that realizes the boundary slope pair $(\frac{p}{q}, \frac{r}{s})$ for any $p, q, r, s \in \mathbb{Z}$? Is it possible to find a quasi-fuchsian surface with boundary slope pair $(\frac{p}{q}, \frac{r}{s})$?

In general it is always possible, through Freedman tubing, to find a surface with boundary slope $\frac{p}{q}$ on one boundary component. This surface by construction is geometrically finite, but clearly has accidental parabolics (essential curves parallel to the boundary). Is it possible to find a quasi Fuchsian surface that accomplishes this?

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