

Surface groups in some surgered manifolds.*

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1 Introduction

An important question in the theory of closed 3-manifolds is whether they are finitely covered by closed manifolds which contain an embedded π_1 -injective surface. An interesting related question is which closed 3-manifolds contain π_1 -injective immersed surfaces.

Suppose that M is an orientable hyperbolic 3-manifold, then $\pi_1(M)$ has a representation as a discrete, faithful, torsion-free subgroup of $PSL_2(\mathbb{C})$. An immersed π_1 -injective surface $g : S \rightarrow M$ determines a subgroup $g_*(\pi_1(S))$ of $\pi_1(M)$, and hence of $PSL_2(\mathbb{C})$. If this subgroup is conjugate into $PSL_2(\mathbb{R})$, then S is a **totally geodesic** (or **Fuchsian**) surface in M . S is said to be **quasi-Fuchsian** if the limit set of a lift of S to the universal cover is topologically a circle.

Work by Marden [Ma] implies that if M is a hyperbolic 3-manifold with boundary and contains a closed π_1 -injective quasi-Fuchsian surface, then this surface will remain π_1 -injective after sufficiently large surgeries. Cooper and Long [CL] have shown recently that if M is a finite volume hyperbolic 3-manifold with a single cusp, then all but finitely many surgeries contain a surface group. In both cases it is known that the surface will remain π_1 -injective after large surgeries, but it does not seem to be known how to compute which surgeries would be large enough. This paper differs in the sense that we will be able to compute which surgeries are large enough.

Let M be a hyperbolic manifold which is the interior of a compact, orientable 3-manifold M^- , with ∂M^- consisting of incompressible tori. Let $\psi : S \rightarrow M$ be map of a closed, orientable, π_1 -injective quasi-Fuchsian surface into M . We will show that for all but finitely many surgeries on any tori in ∂M^- , S remains π_1 -injective in the surgered manifold. Furthermore in the case of totally geodesic surfaces we will compute which surgeries need to be excluded. The main theorem is the following:

Theorem 1.1. *Let M be a complete hyperbolic 3-manifold of finite volume. Let S in M be a closed π_1 -injective quasi-Fuchsian surface. Let Λ be the limit set of S , let $CH(\Lambda)$ be the convex hull of Λ , and let $p : \widetilde{M} \rightarrow M$ be the universal covering map. Let H_1, \dots, H_k be a fixed collection of horoball neighborhoods of the cusps chosen to be disjoint from $p(CH(\Lambda))$. If the p_i/q_i -surgery curves on ∂H_i have representatives which are geodesic in the induced Euclidean metric on ∂H_i and have Euclidean length $> 2\pi$, then S will remain π_1 -injective after surgery.*

In the special case of a totally geodesic surface in a knot or link complement we obtain the following corollary :

Corollary 1.2. *Let M be a hyperbolic knot or link complement. Let S in M be a closed π_1 -injective totally geodesic surface. Let H_1, \dots, H_k be a fixed collection of horoball neighborhoods of the cusps*

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chosen to be disjoint from S . If the p_i/q_i -surgery curves on ∂H_i have representatives which are geodesic in the induced Euclidean metric on ∂H_i and have Euclidean length $> 2\pi$, then S will remain π_1 -injective after surgery.

We will apply the main theorem to the figure-eight knot complement (denoted by M_8). We know that M_8 is an arithmetic manifold. Work by Maclachlan and Reid [MR] implies that M_8 contains infinitely many commensurability classes of closed, immersed, totally geodesic surfaces. After choosing a surface we need to find the largest possible horoball neighborhood of the cusp which does not meet the surface.

We will use the information about the surface and the disjoint horoball neighborhood of the cusp when we apply our main theorem to M_8 and prove the following theorem:

Theorem 1.3. *The figure-eight knot complement contains a π_1 -injective surface, which remains π_1 -injective after all but at most thirteen Dehn Surgeries.*

(These thirteen surgeries are : $\pm p/\pm q = 1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 1/2, 3/2$ and $5/2$)

We will show by direct calculation that all but one of these surgeries give a closed manifold which contains a surface group. The figure-eight knot is the first knot complement for which this is known. This gives the main theorem for the figure-eight knot complement :

Theorem 1.4. *All surgeries, except $1/0$ surgery, on the figure-eight knot complement yield manifolds which contain a surface group.*

We also look at the Whitehead Link complement, call it WL . WL is known to be an arithmetic manifold. $\pi_1(WL)$ is a subgroup of $PSL_2(\mathbb{Z}[i])$. WL is known to contain closed totally geodesic surfaces [MR]. We analyze what happens when we perform Dehn surgery on both components of the Whitehead Link. We obtain the following theorem:

Theorem 1.5. *Let WL be the Whitehead link complement and let $WL(r_1, r_2)$ be the manifold obtained by performing $r_i = p_i/q_i$ Dehn surgery on the i -th torus boundary component ($i=1,2$). There exists a closed π_1 -injective surface in WL which remains π_1 -injective for all but at most sixty surgeries on each link component.*

We use this theorem to deduce that twist knots with a sufficiently large number of twists contain closed, π_1 -injective, immersed surfaces.

Corollary 1.6. *All k -twist knots where $k > 10$ contain a closed, immersed, π_1 -injective surface. Furthermore this surface will remain π_1 -injective after all but at most 60 surgeries.*

Remark: The surfaces constructed in these examples are immersed. Hence we can not yet conclude that the resulting closed manifolds are either Haken or Virtually Haken, but we do know that they contain a surface group.

The organization of the paper is as follows : We will give a short review of Dehn surgery in section 2. In section 3 we will prove the main theorem. In section 4 we will review some results from the theory of arithmetic manifolds, and we will show how to obtain a description of closed totally geodesic surfaces in terms of their lifts to the universal cover. We will use this information in section 5 to determine the size of the horoballs disjoint from the totally geodesic surface. The rest of the paper consists of applications. Section 6 details the results for the figure-eight knot complement, and sections 7 gives the results for the Whitehead Link.

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2 Preliminaries

We can think of surgery on a knot or link complement as truncating the cusp, so that we obtain a compact manifold with torus boundary components, followed by a filling. A p/q Dehn filling on one of the torus boundary components then corresponds to taking a disc D and attaching a neighborhood of the disc, $D \times I$, along its boundary $\partial D \times I$ to a annulus neighborhood of a p/q torus curve. Note that p and q have to be relatively prime. After attaching $D \times I$ in this manner we obtain a manifold with a spherical boundary component, which we cap off to obtain the Dehn filling.

In the particular case of the figure eight-knot complement we know exactly which fillings result in a hyperbolic manifold. In particular we have :

Theorem 2.1. *(Thurston) Every manifold obtained by Dehn surgery along the figure-eight knot has a hyperbolic structure, except the six manifolds :*

$M_8(\pm p/\pm q)$ where $p/q = 1/0, 0/1, 1/1, 2/1, 3/1$ or $4/1$

We have a similar result for surgery on one component of the Whitehead link complement. This result can be found in [NR].

Theorem 2.2. *Every manifold obtained by Dehn surgery on one component of the Whitehead link complement WL has a hyperbolic structure, except the six manifolds $WL(p/q)$, where $\pm(p/q) = 1/0, 0/1, -1/1, -2/1, -3/1$ or $-4/1$*

A lot is already known about manifolds obtained by surgery on the figure-eight knot. If M has finite cover N with $\text{Rank } H_1(N; \mathbb{Z}) > 0$, then M is called **virtually \mathbb{Z} -representable** or **has virtually positive first Betti number**. Theorems due to Hempel, Nicas, Kojima and Long and Baker give us lists of surgeries on the figure-eight knot which give manifolds whose fundamental group is virtually \mathbb{Z} -representable. Much of this information can be found in a paper by M. Baker [B]. In papers by Przytycki [P] and Masters [M] we are given a list of surgeries on the figure-eight knot which result in Virtually Haken Manifolds.

3 Proof of the Main Theorem

Let M_K denote a knot complement, and let M_K^- denote the compact manifold obtained by truncating the cusp. Then the surgered manifold $M(r)$ can be thought of as $M(r) = M_K^- \cup_h V$, where V is a solid torus and $h : \partial V \rightarrow \partial M_K^-$ is the boundary homeomorphism which takes the meridian of ∂V to $p\mu + q\lambda$. If the surgery curve is long enough, we are guaranteed a metric of negative curvature on the resulting manifold. This is known as the 2π theorem of Thurston and Gromov, a proof of which is given in a paper by Bleiler and Hodgson [BH].

Theorem 3.1. *(The 2π -theorem)*

Let M be a complete hyperbolic 3-manifold of finite volume and H_1, \dots, H_k disjoint horoball neighborhoods of the cusps of M . Suppose r_i is a slope on ∂H_i represented by a geodesic α_i with length

in the Euclidean metric satisfying $\text{length}(\alpha_i) > 2\pi$, for each $i = 1, \dots, k$. Then $M(r_1, \dots, r_k)$ has a metric of negative curvature.

To find the length of a p/q -surgery curve on the boundary torus of a compact manifold, we lift a geodesic representative of the curve to the universal cover. The boundary torus will lift to a horosphere at height h . The geodesic representative α of the p/q -curve will lift to a straight line in the horosphere. Suppose a line segment L covers the surgery curve in the boundary torus once. The horosphere has a scaled version of the Euclidean metric and we say that the length of L is the length of the p/q surgery curve in the chosen horoball.

The proof of the 2π theorem is based on the following lemma, which is worth mentioning. We can define a metric on the surgered manifold $M(r) = M_{\bar{K}} \cup V$, so that we have the hyperbolic metric on $M_{\bar{K}}$ and extend this metric to a metric of negative curvature on the solid torus V . This idea will be an important ingredient of the proof of the main theorem in the next section.

Lemma 3.2. *Let V be a solid torus supplied with a hyperbolic metric near its boundary so that ∂V is the quotient of a horosphere. Then the metric near the boundary can be extended to a negatively curved metric on V provided that the length of the Euclidean geodesic representing the meridian curve on ∂V is at least 2π .*

To prove our result, we also need the the following theorem which can be found in do Carmo (see [DC2]).

Theorem 3.3. (Hadamard) *Let M be a complete Riemannian manifold, simply connected, with sectional curvature $K(p, \Sigma) \leq 0$, for all $p \in M$ and for all sections $\Sigma \subset T_p(M)$. Then M is diffeomorphic to \mathbb{R}^n , $n = \dim M$; more precisely $\exp_p : T_p(M) \rightarrow M$ is a diffeomorphism*

Recall that **geodesics** are characterized by the fact that they are locally distance minimizing. The last part of the statement of Hadamard's theorem immediately implies the following corollary:

Corollary 3.4. *Let M be a complete Riemannian manifold. Suppose all sectional curvatures are less than or equal to 0 and $\pi_1(M) = 1$, then there are no closed geodesics.*

Let Γ be any discrete group of orientation preserving isometries of \mathbb{H}^n . If $x \in \mathbb{H}^n$, then the **limit set** $\Lambda_\Gamma \subset S_\infty^{n-1}$ is defined to be the set of accumulation points of the orbits Γ_x of x . Γ is said to be an **elementary group** if the limit set consists of 0, 1 or 2 points. From now on we will always assume that Γ is non-elementary.

Now let S be a surface in a hyperbolic manifold. Let \tilde{S} be the lift of S to the universal cover of M , \tilde{M} . Suppose \tilde{S} meets the sphere at infinity in a set K . Define $CH(K)$ to be the convex hull of K , where the hyperbolic convex hull is the intersection of all hyperbolic half spaces in \mathbb{H}^3 whose intersection with the sphere at infinity contain K . For more detail, see chapter 8 of Thurston's Notes [T].

We may assume that the lift \tilde{S} of a π_1 -injective quasi-Fuchsian surface S is contained in the convex hull. The frontier of the convex hull consists of geodesic segments. Hence this frontier will map down to a pleated surface in the manifold, which is clearly homotopic to the surface we started with. Hence we might as well assume that the lift of the surface to the universal cover is contained in the convex hull.

Lemma 3.5. *Let M be a complete hyperbolic manifold of finite volume. Let S in M be a closed, π_1 -injective quasi-Fuchsian surface. Let Λ be the limit set of $\pi_1(S)$. Let $CH(\Lambda)$ be the convex hull of Λ and let $p : \tilde{M} \rightarrow M$ be the universal covering map. Then $p(CH(\Lambda))$ is compact.*

Proof: S is a closed quasi-Fuchsian surface, hence its fundamental group does not contain any parabolics. This implies that $CH(\Lambda)/\pi_1(S)$ must be compact. Note that $f : \mathbb{H}^3 \rightarrow \mathbb{H}^3/\pi_1(M)$ is a local isometry. Hence so is f restricted to $CH(\Lambda)$. The map $g : CH(\Lambda) \rightarrow CH(\Lambda)/\pi_1(S)$ is a covering map and hence a local isometry. Hence we can define a map $\phi : CH(\Lambda)/\pi_1(S) \rightarrow M$ so that this map is a local isometry. Because $CH(\Lambda)/\pi_1(S)$ is compact, we see that any point x in $p(CH(\Lambda))$ is a bounded distance away from S . This implies that $p(CH(\Lambda))$ must be compact. \square

Corollary 3.6. *Let M be a complete hyperbolic manifold of finite volume. Let S in M be a closed, (possibly immersed,) π_1 -injective quasi-Fuchsian surface. Let Λ be the limit set of $\pi_1(S)$. Let $CH(\Lambda)$ be the convex hull of Λ and let $p : \widetilde{M} \rightarrow M$ be the universal covering map. Then we can choose horoball neighborhoods of the cusp which do not meet $p(CH(\Lambda))$.*

Theorem 3.7. *Let M be a complete hyperbolic 3-manifold of finite volume. Let S in M be a closed, π_1 -injective quasi-Fuchsian surface. Let Λ be the limit set of $\pi_1(S)$, let $CH(\Lambda)$ be the convex hull of Λ , and let $p : \widetilde{M} \rightarrow M$ be the universal covering map. Let H_1, \dots, H_k be a fixed collection of horoball neighborhoods of the cusps chosen to be disjoint from $p(CH(\Lambda))$. If the p_i/q_i -surgery curves on ∂H_i have representatives which are geodesic in the induced Euclidean metric on ∂H_i and have Euclidean length $> 2\pi$, then S will remain π_1 -injective after surgery.*

Proof: As in lemma 3.5 we can define a map $\phi : CH(\Lambda)/\pi_1(S) \rightarrow M$ so that this map is a local isometry. Let $X = CH(\Lambda)/\pi_1(S)$. By hypothesis $p(CH(\Lambda))$ is contained in $M^- = M \setminus (\bigcup_i^k H_i)$. Hence there is a map $\phi^- : X \rightarrow M^-$ so that ϕ^- is a local isometry. There is an inclusion map $i : M^- \rightarrow M(r)$. The 2π -theorem implies that we can put a metric on $M(r)$ so that $\phi_r : X \rightarrow M(r)$, where $\phi_r = i \cdot \phi^-$, is also a local isometry. The map ϕ_r is π_1 -injective. To see this suppose that $\gamma \in \ker(\phi_r)$. Then because of convexity γ is freely homotopic to a geodesic in X . So assume that γ is a geodesic. But then $\phi_r(\gamma)$ is a null homotopic geodesic in $M(r)$. Hence it lifts to a closed geodesic in the universal cover of $M(r)$. This contradicts the corollary to Hadamard's theorem. Hence ϕ_r is π_1 -injective and hence S is a π_1 -injective surface in $M(r)$. \square

In the special case of a totally geodesic surface in a knot or link complement we obtain the following corollary :

Corollary 3.8. *Let M be a hyperbolic knot or link complement. Let S in M be a closed, π_1 -injective totally geodesic surface. Let H_1, \dots, H_k be a collection of horoball neighborhoods of the cusps chosen to be disjoint from S . If the p_i/q_i -surgery curves on ∂H_i have representatives which are geodesic in the induced Euclidean metric on ∂H_i and have Euclidean length $> 2\pi$, then S will remain π_1 -injective after surgery.*

Remark: The 2π -theorem shows that the closed manifolds containing these (immersed) π_1 -injective surfaces have a metric of negative curvature. It has been conjectured that these manifolds are actually hyperbolic manifolds.

4 Totally Geodesic surfaces

4.1 Arithmetic Manifolds

Detailed information about Arithmetic Manifolds can be found in [M], [MR], [R] and [V]. We will now give a summary of the most important ideas.

Recall that a **Kleinian group** is a discrete subgroup of $PSL_2(\mathbb{C})$ and a **Fuchsian group** is a Kleinian group which stabilizes a circle or straight line \mathcal{C} in \mathbb{C} and preserves the components of $\mathbb{C} \setminus \mathcal{C}$.

Let \mathcal{A} be a **quaternion algebra** and \mathcal{O} an **order** in \mathcal{A} . We define \mathcal{O}^1 as follows: $\mathcal{O}^1 = \{x \in \mathcal{O} \mid n(x) = x\bar{x} = 1\}$.

Recall that H and K , groups in X , are **commensurable** if $[H : H \cap K]$ and $[K : H \cap K]$ are both finite.

Assume that Γ , the fundamental group of a 3-manifold M , is a finite covolume group. We say that M is an **arithmetic manifold** if Γ is a subgroup commensurable with $P\rho\mathcal{O}^1$, where we define ρ to be the representation $\rho : \mathcal{O}^1 \rightarrow SL_2\mathbb{C}$, and where P is the projection $P : SL_2\mathbb{C} \rightarrow PSL_2\mathbb{C}$.

Some general results about Kleinian groups and their Fuchsian subgroups can be found in [R]. The results of interest to us are the conditions under which co-compact Kleinian arithmetic groups contain co-compact Fuchsian subgroups.

Theorem 4.1. (Reid) *Let Γ be a co-compact arithmetic Kleinian group. If Γ contains a non-elementary Fuchsian subgroup then Γ contains infinitely many commensurability classes (up to conjugacy in $PSL_2\mathbb{C}$) of co-compact (necessarily arithmetic) Fuchsian subgroups.*

To prove some of the results we need, we have to give some definitions and state some known theorems. (We use [MR] and [LL] as references.) Let \mathcal{O}_d be the ring of integers in $\mathbb{Q}(\sqrt{-d})$, and let Γ_d denote $PSL_2(\mathcal{O}_d)$.

It can be shown that the circle (line) \mathcal{C} has an equation of the form

$$a|z|^2 + Bz + \bar{B}\bar{z} + c = 0, \text{ where } a, c \in \mathbb{Z} \text{ and } B \in \mathcal{O}_d$$

Let $B = \frac{1}{2}(b_1 + b_2\sqrt{-d})$ with $b_i \in \mathbb{Z} (i = 1, 2)$ and $b_1 \equiv b_2 \pmod{2}$ (and $\equiv 0 \pmod{2}$ unless $d \equiv -1 \pmod{4}$). The triple (a, B, c) is called a **primitive triple** if

$$\gcd(a, \frac{b_1}{2}, \frac{b_2}{2}, c) = 1 \text{ for } b_1 \equiv b_2 \equiv 0 \pmod{2}$$

$$\gcd(a, b_1, b_2, c) = 1 \text{ for } b_1 \equiv b_2 \equiv 1 \pmod{2}$$

The **discriminant** of \mathcal{C} is defined to be

$$D = \frac{1}{4}(b_1^2 + db_2^2) - ac$$

Proposition 4.2. (Maclachlan) *Let S be a totally geodesic surface, then the stabilizer of S is :*

$$\text{Stab}(S) = \left\{ \begin{pmatrix} \alpha & D\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in PSL_2(\mathcal{O}_d) \mid |\alpha|^2 - D|\beta|^2 = 1 \right\}$$

where D is the discriminant of the circle at infinity.

An important question at this point is if a group Γ is co-compact or not. The following two theorems will give us a way to determine whether certain arithmetic Kleinian groups are co-compact or not.

We define $A(\Gamma) = \{\sum_{finite} a_i \gamma_i \mid \gamma_i \in \Gamma, a_i \in \mathbb{Q}(tr\Gamma)\}$, where $\mathbb{Q}(tr\Gamma) = \mathbb{Q}(tr\gamma \mid \gamma \in \Gamma)$ is the **Trace Field**

Theorem 4.3. *Let Γ be an arithmetic Kleinian group. Then Γ is co-compact if and only if $\mathcal{A} = A(\Gamma^{(2)})$ is not isomorphic to $M_2(\mathbb{Q}(\sqrt{-d}))$*

Theorem 4.4. *Let \mathcal{A} be a quaternion algebra over K . Then $n_{\mathcal{A}/K}$ is isotropic (i.e. $\exists x \neq 0 \in \mathcal{A}$, so that $n(x) = 0$) if and only if \mathcal{A} is isomorphic to $M_2(K)$.*

4.2 Totally Geodesic Surfaces in the Figure-Eight Knot Complement

Let K_8 be the figure-eight knot and let M_8 be $S^3 \setminus N(K_8)$.

Then $\pi_1(M_8) = \langle a, b | b^{-1}aba^{-1}b = ab^{-1}aba^{-1} \rangle$, where we can choose

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 0 \\ \omega + 1 & 1 \end{pmatrix}$$

with $\omega^3 = 1, \omega \neq 1$ (for a reference see [Ri]).

A totally geodesic surface in M_8 lifts to a plane in \mathbb{H}^3 . Suppose S_D lifts to a plane P_D , where $P_D \cap S_\infty^2$ is a round circle of radius \sqrt{D} , centered at the origin. We have the following theorem (see Maclachlan [M])

Theorem 4.5. (Maclachlan) *The quaternion algebra associated to the Fuchsian subgroup of Γ_d of discriminant D is isomorphic to $\left(\frac{-d, D}{\mathbb{Q}}\right)$*

We have a list of possible surfaces to work with, via the following proposition :

Proposition 4.6. *Let M_8 be the figure-eight knot complement. Let P_D be a plane in \mathbb{H}^3 , where $P_D \cap S_\infty^2$ is a round circle of radius \sqrt{D} , centered at the origin. If $D \equiv 2 \pmod{3}$, then P_D projects down to a closed, immersed, totally geodesic surface S_D in M_8*

Proof: D is a square free number. Hence the equation $|\alpha|^2 - D|\beta|^2 = 1$ has infinitely many integral solutions. (This is Pell's Equation.) This implies that $Stab(S_D)$ is non-elementary. By the previous theorem the quaternion algebra \mathcal{A} associated to the Fuchsian subgroup of Γ_3 of discriminant D is isomorphic to $\left(\frac{-3, D}{\mathbb{Q}}\right)$. Hence a typical element x from \mathcal{A} has the following general form:

$x = x_0 + x_1i + x_2j + x_3ij$, with $x_i \in \mathbb{Q}, i^2 = -3, j^2 = D$ and $ij = -ji$. The norm of x , $n(x)$, is given by $n(x) = x\bar{x}$. Note that when we consider the equation $x\bar{x} = 0$ we actually can assume that $x_i \in \mathbb{Z}$ (multiply through by the appropriate least common multiple of the denominators of the x_i).

It is easy to check that the normform $n_{\mathcal{A}/K}$ is not isotropic :

Suppose $n_{\mathcal{A}/K}(x) = x_0^2 + 3x_1^2 - Dx_2^2 + 3Dx_3^2 = 0$ and $x \neq 0$

$D \equiv 2 \pmod{3}$ implies that $x_0^2 - 2x_2^2 \equiv 0 \pmod{3}$. Hence $x_0^2 \equiv x_2^2 \equiv 0 \pmod{3}$.

This implies that $x_0^2 = 9\bar{x}_0^2$ and $x_2^2 = 9\bar{x}_2^2$ and we obtain obtain the expression $3\bar{x}_0^2 + x_1^2 - 3D\bar{x}_2^2 + Dx_3^2 = 0$. This now implies that $x_1^2 - 2x_3^2 \equiv 0 \pmod{3}$. By a similar argument we get $\bar{x}_0^2 + 3\bar{x}_1^2 - D\bar{x}_2^2 + 3D\bar{x}_3^2 = 0$. Clearly we eventually reach a contradiction. Hence $n_{\mathcal{A}/K}$ is not isotropic and hence by theorem 4.4 the quaternion algebra \mathcal{A} is not isomorphic to a matrix algebra. This implies by theorem 4.3 that Γ is a co-compact group. It also follows from construction that S_D is immersed and totally geodesic (and hence incompressible). \square

Corollary 4.7. *Let M_8 be the figure-eight knot complement. Let P_2 be a plane in \mathbb{H}^3 , where $P_2 \cap S_\infty^2$ is a round circle of radius $\sqrt{2}$, centered at the origin. Then there exists a closed, orientable, π_1 -injective, totally geodesic surface in M_8 , call it S_2 , which lifts to P_2 .*

Proof: $D = 2$, hence the previous proposition implies that P_2 maps down to a closed, immersed, totally geodesic surface. We may assume that the surface is orientable. For if it is not, we can take the orientation double cover and obtain an orientable surface. The surface is totally geodesic by construction and hence also π_1 -injective. \square

5 Determining the size of the horoball neighborhood of the cusp

When performing Dehn surgery, we want to truncate the cusps so that we obtain a compact manifold with torus boundary components disjoint from the projection of the convex hull of the quasi-Fuchsian surface.

We can show that finding such a horoball neighborhood of the cusp is always possible (see lemma 3.5 and corollary 3.6), but computing the size is difficult. In the case that the surface is totally geodesic and the fundamental group of the manifold is a subgroup of a Bianchi group, we can determine precisely when the horoball neighborhood of the cusp is disjoint from the surface. Intuitively we need to determine how far we can lower the horosphere in the universal cover without meeting the lift of the totally geodesic surfaces to \mathbb{H}^3 .

Recall the plane we called P_D , meets the sphere at infinity in a round circle, of (euclidean) radius \sqrt{D} , centered at the origin of the deck in the upper half-space model. To compute the size of the horoball neighborhood of the cusp disjoint from the totally geodesic surface, we need to understand how transformations act on P_D . The lifts of the totally geodesic surface to \mathbb{H}^3 have invariants associated to them. The planes can be described in terms of primitive triples and discriminants (See section 4.1 for definitions).

There is a lemma in [MR] which gives us some general facts about these primitive triples and discriminants:

Lemma 5.1. *Let $\mathcal{C}, \mathcal{C}'$ be represented by triples (a, B, c) and (a', B', c') respectively and let $T\mathcal{C} = \mathcal{C}'$ with $T \in \Gamma_d$, T acts as a linear fractional transformation on the triples.*

1. (a, B, c) is primitive if and only if (a', B', c') is primitive.
2. If (a, B, c) is primitive, then $D(\mathcal{C}) = \mathcal{D}(\mathcal{C}')$

Proof: Define $\Sigma_d = \{\text{circles } \mathcal{C} \text{ represented by primitive triples in } \mathcal{O}_d\}$. Let \mathcal{C} be represented by the primitive triple (a, B, c) . Define $\Phi(a, B, c) = \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix}$.

Φ defines a bijection from Σ_d to

$$\mathcal{H}_d = \left\{ \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} \mid a, c \in \mathbb{Z}, B \in \mathcal{O}_d, ac - |B|^2 = -D, (a, B, c) \text{ is primitive} \right\}$$

$T \in \Gamma_d$ acts on \mathcal{H}_d by the following action :

$$\begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} \longrightarrow T \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} T^*, \text{ where } T^* \text{ is the Hermitian of } T$$

Proof of (1) : Suppose (a, B, c) is primitive (assume $b_1 \equiv b_2 \equiv 0 \pmod{2}$, the argument when $b_1 \equiv b_2 \equiv 1 \pmod{2}$ is similar). If T acts on $\Phi(a, B, c)$ and $\Phi(a', B', c')$ is not primitive, then we can write

$$T \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} T^* = \begin{pmatrix} a' & B' \\ \bar{B}' & c' \end{pmatrix} = p^2 \begin{pmatrix} a'' & B'' \\ \bar{B}'' & c'' \end{pmatrix} \text{ for some prime } p.$$

$$\text{But we know that } T^{-1} \begin{pmatrix} a' & B' \\ \bar{B}' & c' \end{pmatrix} T^{-1*} = p^2 T^{-1} \begin{pmatrix} a'' & B'' \\ \bar{B}'' & c'' \end{pmatrix} T^{-1*} = \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix}$$

This would imply that (a, B, c) was not primitive, which is a contradiction. Hence (a', B', c') must be primitive as well.

Proof of part (2) : This follows from the fact that $\text{Det}\Phi(a, B, c) = \text{Det}\Phi(a', B', c') = -D$.

□

Note that the hemisphere P_D has Euclidean radius \sqrt{D} . The lemma above implies the following result:

Lemma 5.2. *Let P_D be a plane in \mathbb{H}^3 which meets the sphere at infinity in a circle of radius \sqrt{D} centered at the origin and projects down to a closed surface. Let $\Gamma_d = \text{PSL}_2(\mathcal{O}_d)$. Then any translate of P_D under an element of Γ_d is a hemisphere with a radius of at most \sqrt{D} .*

Proof: P_D meets the sphere at infinity in a circle \mathcal{C} . The triple associated with this sphere is $(1, 0, -D)$. This triple is clearly primitive. Any $g \in \Gamma_d$ will map this triple to another primitive triple with discriminant D by the previous lemma. A direct computation shows that the radius of this new hemisphere is $\sqrt{\frac{D}{(a')^2}}$ if we assume that a' is not zero. This shows that the maximal height occurs when $a' = \pm 1$ and the maximal height attained is \sqrt{D} .

Now assume that $a' = 0$. Then after translation, the circle passes through infinity. There is a parabolic element which causes the original circle to intersect itself. Apply this parabolic to the translated circle. Then we must see a lift of a double curve with one end at infinity. Infinity is a fixed point, hence some power of this parabolic must map the lift of the double curve to itself. This implies that the fundamental group of the surface contains a parabolic. But closed surfaces do not contain parabolics. Hence $a' \neq 0$.

□

This lemma shows directly that given the surface S_2 which lifts to P_2 , we can remove a horoball neighborhood of the cusp whose torus boundary lifts to the horosphere given by $z = \sqrt{2} + \epsilon$ ($\epsilon > 0$).

Lemma 5.3. *Let M be the figure-eight knot complement. Let P be the horoball neighborhood of the cusp chosen so that the boundary ∂P lifts to the horosphere with fixed point at infinity and at height $\sqrt{2} + \epsilon$ ($\epsilon > 0$). Then the boundary of the projection of P down to M_8 is disjoint from S_2 .*

6 Application I : the Figure-Eight Knot Complement

6.1 Incompressibility of S_2 after Surgery

Using the notation from Thurston's Notes [T], we can identify the closure of M_8 with the complement of an open tubular neighborhood of the figure-eight knot, K_8 , in S^3 . We can choose generators μ and λ for $H_1(\partial M_8)$. There is a standard way to do this so that μ is the meridian (it bounds a disk in the solid torus around K_8), and λ is a longitude (it is homologous to zero in M_8). It can be shown (see [T], chapter 4) that the meridian and the longitude of the boundary torus are given by :

$$\mu = y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \lambda = x + 2y = \begin{pmatrix} 1 & 4\omega + 2 \\ 0 & 1 \end{pmatrix}.$$

where $\omega^3 = 1$. Hence the length of the lift of the surgery curve for (p, q) -surgery to the horosphere at height $\sqrt{2}$ is given by :

$$L = \frac{1}{\sqrt{2}}(p^2|\mu|^2 + q^2|\lambda|^2)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}(p^2 + 12q^2)^{\frac{1}{2}}$$

We can use the main theorem to compute which Dehn surgeries on the figure-eight-knot complement will leave the surface S_2 π_1 -injective. Note that due to the high degree of symmetry of the figure-eight knot it follows that $(M_8)_{(p,q)} = (M_8)_{(-p,q)}$.

Theorem 6.1. *The figure-eight knot complement contains a π_1 -injective surface, which remains π_1 -injective after all but at most thirteen Dehn Surgeries.*

(These thirteen surgeries are : $\pm p/\pm q = 1/0, 0/1, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 7/1, 8/1, 1/2, 3/2$ and $5/2$)

Proof: It follows from corollary 4.7 that there exists an orientable, closed, totally geodesic surface which lifts to the plane P_2 . By lemma 5.3 we have that for all $\epsilon > 0$ the horoball of radius $\sqrt{2} + \epsilon$ which is disjoint the surface S_2 . Hence we can remove this horoball, and using theorem 3.7 we can do surgery resulting in a manifold with negative curvature containing S_2 as an π_1 -injective surface if the surgery curve has length greater than 2π . For the figure-8-knot we need to know when $\frac{1}{\sqrt{2}+\epsilon}(p^2|\mu|^2 + q^2|\lambda|^2) > 2\pi$

In \mathbb{R}^2 , $\mu = (1, 0)$ and $\lambda = (0, 2\sqrt{3})$. Hence our length requirement becomes :

$$(p^2 + 12q^2)/(\sqrt{2} + \epsilon)^2 > 4\pi^2$$

Suppose that $\epsilon < \frac{1}{10}$. If $q = 1$, then $p^2 > (\sqrt{2} + \epsilon)^2 4\pi^2 - 12 (\approx 79)$ implies that we want $|p| > 8$.

If $q = 2$, then $p^2 > (\sqrt{2} + \epsilon)^2 4\pi^2 - 48 (\approx 43)$ implies that we want $|p| > 6$.

For $q \geq 3$, p can be any number.

By theorem 3.7 the surface remains π_1 -injective as long as we do not perform one of these thirteen surgeries. □

6.2 The Exceptional Surgeries

Using the computer program GAP all but one of the exceptional surgeries can be dealt with. This results in the following theorem :

Theorem 6.2. *All surgeries on the figure-eight knot complement, except 1/0 surgery, give manifolds which contain a surface group.*

$M_8(8/1)$: Contains a subgroup of index 2 (Homology \mathbb{Z}_{20}), which contains a subgroup of index 5 with homology $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$.

$M_8(7/1)$: There is a subgroup of index 7 (homology $\mathbb{Z}_{14} \oplus \mathbb{Z}_{14}$), with a subgroup of index 7 containing $\mathbb{Z} \oplus \mathbb{Z}$.

$M_8(6/1)$: There is an index 10 subgroup with homology $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

$M_8(5/1)$: There are 2 subgroups of index 7 (both with homology \mathbb{Z}_{10}) and (at least) one of them has a subgroup of index 6 containing $\mathbb{Z} \oplus \mathbb{Z} \oplus G$, where G is a finite group.

$M_8(1/2)$: There is a unique subgroup of index 8 which has homology $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. This subgroup has a commutator subgroup with homology $\bigoplus_{i=1}^{16} \mathbb{Z} \oplus G$, where G is a torsion group.

$M_8(3/2)$: Has a unique subgroup of index 3. This subgroup has a subgroup of index 8 with homology $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$.

$M_8(5/2)$: There is a unique subgroup of index 5. This subgroup has a subgroup of index 7, with homology $\mathbb{Z} \oplus H$, where H is some non-trivial group.

Hence these closed manifolds are virtually \mathbb{Z} -representable (have virtually positive first Betti number).

$M_8(0/1)$ is a Solv manifold and is known to contain a surface group.

$M_8(1/1)$, $M_8(2/1)$ and $M_8(3/1)$ are Small Seifert Fibered spaces, i.e. Seifert Fibered Spaces which fiber over the 2-sphere and have at most 3 exceptional fibers.

$M_8(4/1)$ is a Toroidal manifold.

This shows that the surgeries on the figure-eight-knot (with the exception of $M_8(1/0)$) give manifolds which contain surface groups.

We know that $1/0$ surgery gives the 3-sphere, which does not contain any surface groups.

7 Application II : the Whitehead Link

7.1 Surgery on Both Components of the Whitehead Link

We can compute the Wirtinger representation of the first fundamental group of the Whitehead Link (WL), which looks as follows :

$$\pi_1(WL) = \langle x_1, x_5 | x_1 x_5 x_1^{-1} x_5 x_1 x_5^{-1} x_1 x_5 = x_5 x_1 x_5^{-1} x_1 x_5 x_1^{-1} x_5 x_1 \rangle$$

Using results from Maclachlan and Reid [MR], we can show that there is a closed totally geodesic surface in the Whitehead Link which lifts to a plane, P_3 , which traces out a circle of radius $\sqrt{3}$ in the universal cover \mathbb{H}^3 . We need the following lemma from [MR] :

Lemma 7.1. *Let $\mathcal{C} \in \Sigma_d = \{\text{circles } \mathcal{C} \text{ represented by primitive triples in } \mathcal{O}_d\}$. $D(\mathcal{C}) = D$. Then $\text{Stab}(\mathcal{C}, \Gamma_1)$ is co-compact if and only if D is divisible by an odd power of a prime $\equiv 3 \pmod{4}$*

This lemma allows us to show that the Whitehead Link complement contains a closed totally geodesic surface which we can describe precisely.

Corollary 7.2. *P_3 covers a closed, orientable, π_1 -injective, immersed, totally geodesic surface, S_3 , in the Whitehead link.*

Proof: $D = 3$ implies via the previous lemma that $\text{Stab}(\mathcal{C}_3, \Gamma_1)$ is co-compact. Hence P_3 will cover a closed surface in WL, call it S_3 , and we may assume it is orientable. It is totally geodesic by construction and hence π_1 -injective. □

Lemma 7.3. *Let WL be the Whitehead Link. For all $\epsilon > 0$ the horosphere at height $\sqrt{3} + \epsilon$ maps down to a boundary torus disjoint from S_3 .*

Proof: This follows immediately from lemma 5.2. □

This implies the following theorem :

Theorem 7.4. *Let WL be the Whitehead link complement and let $WL(r_1, r_2)$ be the manifold obtained by performing $r_i = p_i/q_i$ Dehn surgery on the i -th torus boundary component ($i=1,2$). There exists a closed π_1 -injective surface in WL which remains π_1 -injective for all but at most sixty surgeries on each link component.*

Proof: We have two cusps in the Whitehead link complement. The meridian of one of the cusps is represented by the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which has infinity as its fixed point. Hence truncating the cusp corresponds to removing a horoball with its fixed point at infinity in the universal cover. The meridian of the other cusp is given by $\begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix}$ which has its fixed point at zero. Truncating this cusp corresponds to removing a horoball with fixed point at zero in the universal cover. If we think of excising the horoball as taking a horosphere with the same fixed point and lowering it as much as possible, it is clear that we have to compute how far we can lower both horospheres without meeting each other or the lifted surface P_3 . It is clear that P_3 is determined by $|z|^2 - 3 = 0$ which has $(1, 0, -3)$ as its primitive triple and $D = 3$. By lemma 5.2 we see that for all $\epsilon > 0$ the horoball at height $\sqrt{3} + \epsilon$ will be disjoint from P_3 and its translates. If we apply the map $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then we switch the roles of zero and infinity and we map the plane $|z|^2 - 3 = 0$ to $3|z|^2 - 1 = 0$. This new plane still has $D = 3$, so that any element of the fundamental group of the Whitehead link complement will move the surface away from a horosphere at height $\sqrt{3} + \epsilon$. This implies that the surgery conditions are the same for both cusps. Note that the representations of the peripheral groups tell us that we may choose the meridian to have length 1 and the longitude to have length 2. This gives $p_i^2 + 4q_i^2 > 3(2\pi)^2$ as the condition for surgery. This gives the sixty excluded surgeries as mentioned in the theorem. \square

7.2 Twist Knots

If we consider the manifolds obtained by Dehn surgery on just one component of the Whitehead link and denote the resulting manifold by $WL_{(p,q)}$, then a proposition from a paper by Hodgson, Meyerhoff and Weeks [HMW] gives us the following information:

Proposition 7.5. *$WL(p/q)$ and $WL(p'/q')$ are homeomorphic if and only if $(p/q) = \pm(p'/q')$*

When we perform $(1, n)$ surgery on an unknotted components of a link, we obtain a knot or link complement. To see this note that if one of the link components is unknotted, then this implies that we can view $1/n$ surgery on that component as performing n meridional twists on the complimentary solid torus followed by trivially filling in the unknotted component. (See Chapter 9H in Rolfsen [Ro].) This shows that we will obtain a knot complement in S^3 . $1/n$ surgery on one of the components of the Whitehead link results in a twist knot. Given the correct choice of generators we obtain a $2n$ -twist knot after $1/n$ surgery when $n > 0$ and a $2n - 1$ -twist knot if $n < 0$. It was shown in [R] that infinitely many twist knots cannot contain closed, immersed, totally geodesic surfaces.

Theorem 7.6. *(Reid) There exist infinitely many twist knots which contain no closed totally geodesic surfaces and exactly one commensurability class of non-closed totally geodesic surface.*

It is fairly easy to see that $1/n$ surgery does not change the surgery description on the remaining component [Ro]. This implies that we can apply the arguments from theorem 7.4 out of the previous

section again to show that even though this surface in the twist knots is not totally geodesic, it will remain π_1 -injective after all but 60 surgeries.

We have the following corollary :

Corollary 7.7. *All k -twist knots where $k > 10$ contain a closed, immersed, π_1 -injective surface. Furthermore this surface remains π_1 -injective except for at most sixty surgeries.*

Proof: As before, there exists a closed, immersed, π_1 -injective totally geodesic surface in the Whitehead Link complement which lifts to the plane P_3 . Using the same methods as before, we remove a horoball neighborhood of the cusp which lifts to a horoball with fixed point at infinity and horosphere boundary at height $\sqrt{3}$. Note that due to the symmetry of the Whitehead Link we need not take into consideration which component we are filling. Up to homeomorphism we may assume we are filling the component which corresponds to the vertex at infinity. Restricting ourselves to $(1, n)$ surgery we obtain as our condition for surgery :

$$1 + 4n^2 > 3(2\pi)^2$$

Hence we need to exclude $|n| \leq 5$. But a $1/n$ surgery results in $2n$ extra crossings. Hence for a k -twist knots we need $k > 10$. To see that the surface in the twist knot remains injective after surgery, think of p/q surgery on the twist-knot as surgery on both components of the Whitehead Link, where one component has surgery coefficients p/q and the other has surgery coefficients $1/n$. By the previous theorem the resulting manifolds are homeomorphic. Hence the result follows immediately from Theorem 7.4. □

Remark : Similar results hold for the Borromean Rings. This link complement contains a closed, immersed, π_1 -injective surface which remains π_1 -injective after all but at most 120 surgeries on each cusp. We can also show that there is an infinite family of knots (belonging to 2-bridge knots) and an infinite family of links containing a closed π_1 -injective surface which will remain π_1 -injective after all but 120 surgeries on each remaining cusp.

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