Section 19 – The Product Topology

We will consider these products of topological spaces :
The finite Cartesian product $X_1 \times X_2 \times ... \times X_n$
The infinite Cartesian Product $X_1 \times X_2 \times X_3 \times ...$

There are two topologies: the box topology and the product topology.
Basis for ***the box topology***: sets of the form $U_1 \times ... \times U_n$ ($U_1 \times U_2 \times ...$ resp.)
Sub-Basis for ***the product topology***: sets of the form $\pi_1^{-1}(U_i)$

These topologies are the same for the finite products, but they do not agree on the infinite products.
The product topology is more commonly used.

Convention: When considering a product space, we shall assume it is given the product topology unless stated otherwise.

Some Definitions

Let J be an index set. Given a set X, we define ***a J-tuple of elements of X*** to be a function $\mathbf{x}: J \to X$. We will often write $x(\alpha) = x_\alpha$; we call it the α th coordinate of x. The function itself can also be denoted by $(x_\alpha)_{\alpha \in J}$
The set of all J-tuples of elements of X is denoted by X^J

Let $(A_\alpha)_{\alpha \in J}$ be an indexed family of sets; Let $X = \bigcup_{\alpha \in J} A_\alpha$. ***The Cartesian product*** of this indexed family, denoted by $\prod_{\alpha \in J} A_\alpha$ is defined to the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$.

If the sets A_α are all equal to X, then the product is just the set X^J of all J-tuples of X.

The Box Topology: For the indexed family of topological spaces $X = \bigcup_{\alpha \in J} A_\alpha$ The basis consisting of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where U_α is open in Xα generates the box topology.

We define ***the projection map*** $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ to be the function assigning to each element of the product space itsβth coordinate $\pi_\beta\left((x_\alpha)_{\alpha \in J}\right) = x_\beta$

Theorem 19.1 – Comparison of the box and product topologies.
The box topology on $\prod X_\alpha$ has as a basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α.
The product topology ….