Section 18 Continuous Functions.

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open set V in Y, the set $f^{-1}(V)$ is open in X.

Note that continuity depends on the function f, the spaces X and Y as well as the topologies defined on X and Y.

Fact:
It suffices to check continuity on a basis or sub-basis of a topology.
The ε-δ definition for continuity is equivalent to our definition.
The ε-δ definition and convergent sequence definition for continuity do not generalize to general spaces.

Theorem 18.1: Let X and Y be topological spaces; let $f : X \to Y$ TFAE
(1) f is continuous
(2) For every subset A of X we have $f(\overline{A}) \subset \overline{f(A)}$
(3) For every closed set B of Y the set $f^{-1}(B)$ is closed in X.
(4) For each $x \in X$ and each neighborhood V of f(x), there is a neighborhood U of x so that $f(U) \subset V$

Proof: *Show* $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ *and* $(1) \Rightarrow (4) \Rightarrow (1)$

We say **f is continuous at a point x** is property (4) holds at x.

Homeomorphisms

The function $f : X \to Y$ is called a **homeomorphism** is f is bijective and if both f and its inverse are continuous.

A homeomorphism gives us a bijective correspondence between X and Y as well as between the collections of open sets. I.e. there is a correspondence between their topologies as well.

A topological property of X is a property expressed in terms of the topology of X (i.e. in terms of the open sets).

A homeomorphism will preserve topological properties.

The map $f: X \to Y$ is called a **(topological) imbedding** of X into Y is f is an injective continuous map and if the restriction $f: X \to f(X) \subset Y$ is a homeomorphism.

Look at examples 4-7 on pages 106-107.

Constructing Continuous Functions

Theorem 18.2 Let X, Y, and Z be topological spaces.
(1) The constant function is continuous.
(2) The inclusion map is continuous.
(3) The composition of two continuous maps is continuous.
(4) If f is continuous on X then it will also be continuous when restricted to a subset of X.
(5) If the range can be expanded or restricted then those functions will be continuous.

Theorem 18.3 – The Pasting Lemma
Let $X = A \cup B$, where A and B are closed in X. Let $f: A \to Y$ and $g: B \to Y$ be continuous. If f(x) = g(x) for every $x \in A \cap B$ then f and g combine to give a continuous function $h: X \to Y$ defined by setting h(x) = f(x) when $x \in A$ and h(x) = g(x) when $x \in B$

Theorem 18.4 – Maps into Products
Let $f: A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$
Then f is continuous if and only if the coordinate functions $f_1: A \to X$ and $f_2: A \to Y$ are continuous.