

## Section 17 Closed Sets and Limit Points

**Definition:** A subset  $A$  of a topological space  $X$  is said to be **closed** if  $X-A$  is open

**Theorem 17.1:** Let  $X$  be a topological space. Then the following hold:

- (1)  $\emptyset$  and  $X$  are closed
  - (2) Arbitrary intersections of closed sets are closed.
  - (3) Finite unions of closed sets are closed.
- (Proof: from definitions)

**Theorem 17.2:** Let  $Y$  be a subspace of  $X$ . Then  $A$  is closed in  $Y$  iff  $A$  equals the intersection of a closed set of  $X$  with  $Y$ .

(Proof: both directions is straight forward).

**Theorem 17.3:** Let  $Y$  be a subspace of  $X$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$ , then  $A$  is closed in  $X$ .

**Definition:** The **interior of a set**  $A \subset X$  is the union of all open sets contained in  $A$ .

**Definition:** The **closure of a set**  $A \subset X$  is the intersection of all closed sets containing  $A$ .

Note:  $\text{Int}(A) \subset A \subset \text{Cl}(A)$

**Theorem 17.4:** Let  $Y$  be a subspace of  $X$ ; let  $A$  be a subset of  $Y$ ; let  $\bar{A}$  denote the closure of  $A$  in  $X$ . Then the closure of  $A$  in  $Y$  equals  $\bar{A} \cap Y$

**Definition:** We shall say that a set  $A$  **intersects** a set  $B$  if the intersection  $A \cap B$  is non-empty.

**Theorem 17.5:** Let  $A$  be a subset of the topological space  $X$ .

Then  $x \in \bar{A}$  iff every open set  $U$  containing  $x$  intersects  $A$ .

Supposing the topology of  $X$  is given by a basis, then  $x \in \bar{A}$  iff every basis element  $B$  containing  $x$  intersects  $A$ .