Section 16 The Subspace Topology.

Definition: Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, then the collection $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$ is a topology on Y. This topology is called the **subspace topology**.

Lemma 16.1: If $\mathcal{B}$ is a basis for the topology of X, then the collection $\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.
(Proof: use lemma 13.2 – straight forward)

Remark: When dealing with a space X and its subspace Y, we may need to specify *where* our sets are open. Given a set U we would say "U is open in Y (or open relative to Y)" or "U is open in X", to specify if U belongs to the subspace topology of Y or to the topology of X.

Lemma 16.2: Let Y be a subspace of X. If U is open in Y and Y is open in U, then U is open in X.
(proof: defn.)

Theorem 16.3: If $A \subset X$ and $B \subset Y$, then the product topology on $A \times B$ is the same as the subspace topology that $A \times B$ inherits from $X \times Y$.
(Proof: show they have the same basis.)

Examples: Compare and contrast the subspace topology and the order topology on a subset Y of $\mathbb{R}$.

Definition: Let X be an ordered set. $Y \subset X$ is **convex** if for each pair of points a < b of Y, the entire interval (a,b) of points in X is contained in Y.

Theorem 16.4: Let X be an ordered set in the order topology; Ley Y be a subset of X that is convex in X. Then the order topology on Y is the same as the subspace topology that Y inherits from X.