Section 13 Basis for a Topology

Two important properties of bases are:
1. The base elements cover X. (For each x in X there is at least one basis element B in $\mathscr{B}$ containing x.)
2. Let B_1, B_2 be base elements and let I be their intersection. Then for each x in I, there is another base element B_3 containing x and contained in I.

If $\mathscr{B}$ satisfies both of the conditions 1 and 2, then there is a unique topology on X for which B is a base; it is called the topology generated by B. (This topology is the intersection of all topologies on X containing B.)

Munkres defines ***a topology T generated by*** $\mathscr{B}$ as follows:
A subset U of X is open in X (i.e. an element of T) if for all x in U there is a B in $\mathscr{B}$ so that x is in B and B is contained in U.
[$U \subset X$ is open if $\forall x \in U, \exists B \in \mathscr{B}$ so that $x \in B$ and $B \subset U$]
Note that each basis element is an element of T.

Some examples:
1. $\mathscr{B}$ = collection of all circular regions in the plane. (Check properties of basis).
2. $\mathscr{B}$ = collection of all rectangular regions in the plane. (Check properties of basis).
3. $\mathscr{B}$ = collection of all one point subsets of X. (Check properties of basis).
4. $\mathscr{B}$ = collection of all open sets (a,b) of the real line – ***Standard or Usual Topology***.
5. $\mathscr{B}$ = collection of all half-open sets [a,b) of the real line – ***Lower Limit Topology***.
6. $\mathscr{B}$ = collection of all open sets ((a,b) – K) of the real line, where K is the set of all numbers 1/n for n in the positive integers – ***K – Topology*** on the real line.

Proposition: The collection T generated by $\mathscr{B}$ is a topology.

Lemma 13.1. Let X be a set; let $\mathscr{B}$ be a basis for a topology T on X. Then T equals the collection of all unions of elements of $\mathscr{B}$.

Lemma 13.3 Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for T and T′, resp., on X. The TFAE:
(1) T′ is finer than T.
(2) For each x $\in$ X and each B in $\mathscr{B}$ containing x, there is a basis element B′ $\in \mathscr{B}'$ such that x $\in$ B′ $\subset$ B

Lemma 1.3.4 The Lower Limit topology and the K-topology are strictly finer than the standard topology on the real line, but are not comparable with one another.