restart:
with(plots) : with(plottools) :

Each time you restart this worksheet, remember to execute the command above (this calls up the complicated plotting tools). You are expected to execute all of the Maple commands and to answer all of the questions in the blue font.

Solids of Intersection: Visualization and Volumes

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Many physical shapes can be described as the intersections of two (or several) 3-dimensional surfaces. Given a set of surfaces, however, it can be very difficult to visualize the intersection. Once you can imagine the solid of intersection, it still may be difficult to find equations for the intersection…and even harder to convince yourself you're right!

In this worksheet, we'll use Maple to help us visualize solids of intersection and to confirm our observations about these solids. Once we understand solids of intersection, we'll use integration techniques to find volumes (and maybe surface areas?????) of these solids.

Solid #1: The Intersection of Two Cylinders

It can be very difficult to visualize the intersection of two cylinders (and even harder to draw them by hand!). Such graphs are an excellent task for Maple. Let's plot the intersection of two circular cylinders.

graph1 := implicitplot3d\( \left(x^2 + y^2 = 8, x = -3 .. 3, y = -3 .. 3, z = -3 .. 3, \text{axes = boxed, style = surface, color = red, transparency = .2} \right) \):

graph2 := implicitplot3d\( \left(y^2 + z^2 = 8, x = -3 .. 3, y = -3 .. 3, z = -3 .. 3, \text{axes = boxed, style = surface, color = blue} \right) \):

display( \{graph1, graph2\});
In order to eventually set up an integral to compute the volume of the solid where these two cylinders intersect, it will be valuable to visualize this solid.

The boundaries of intersection between the two cylinders are the ellipses where the cylinders intersect the planes $x = z$ and $x = -z$.

```plaintext
plane1 := implicitplot3d(x = z, x = -3 .. 3, y = -3 .. 3, z = -3 .. 3, color = yellow, style = surface) :
plane2 := implicitplot3d(x = z, x = -3 .. 3, y = -3 .. 3, z = -3 .. 3, color = yellow, style = surface) :
plane3 := implicitplot3d(x = -z, x = -3 .. 3, y = -3 .. 3, z = -3 .. 3, color = yellow, style = surface) :
display({graph2, plane1, graph1, plane2});
```
These can be described parametrically. It is not hard to see that the desired parametric equations are

\[ x = \sqrt{8} \cos(t) \]
\[ y = \sqrt{8} \sin(t) \]
\[ z = \sqrt{8} \cos(t) \]

and

\[ x = \sqrt{8} \cos(t) \]
\[ y = \sqrt{8} \sin(t) \]
\[ z = -\sqrt{8} \cos(t) \]

(1.1)

We now add these to the cylinders (and remove the planes)

curve1 := spacecurve([sqrt(8) \cdot \cos(t), sqrt(8) \cdot \sin(t), sqrt(8) \cdot \cos(t)], s = -10..10, t = 0..2 \cdot \pi, thickness = 4, color = black) :
curve2 := spacecurve([sqrt(8) \cdot \cos(t), sqrt(8) \cdot \sin(t), -sqrt(8) \cdot \cos(t)], s = -10..10, t = 0..2 \cdot \pi, thickness = 4, color = black, axes = boxed) ;
The region is, therefore, four "orange-slice" type pieces, lying on the cylinders. With a little bit of effort, we can plot the outline of the solid whose volume we seek and dispense with the initial cylinders.

\[
\text{top} := \text{plot3d}(\sqrt{8-y^2}, x=-\sqrt{8}..\sqrt{8}, y=-\sqrt{8-x^2}..\sqrt{8-x^2}, \text{view}=-3..3, \text{color} = \text{grey}, \text{transparency} = .6) ;
\]

\[
\text{bottom} := \text{plot3d}(-\sqrt{8-y^2}, x=-\sqrt{8}..\sqrt{8}, y=-\sqrt{8-x^2}..\sqrt{8-x^2}, \text{view}=-3..3, \text{color} = \text{grey}, \text{transparency} = .6) ;
\]

\[
\text{front} := \text{rotate}\left(\text{bottom}, 0, \frac{\pi}{2}, 0\right) ;
\]

\[
\text{back} := \text{rotate}\left(\text{bottom}, 0, -\frac{\pi}{2}, 0\right) ;
\]

\[
\text{display}(\{\text{top, bottom, front, back, curve1, curve2}\}) ;
\]
Now that we have a decent visualization of the solid, we set out to devise a strategy for finding its volume.

For this, we will use a method from one-variable calculus called the \textit{method of cross-sections}.

If we slice the solid perpendicular to the \(y\)-axis (for \(-\sqrt{8} \leq y \leq \sqrt{8}\)), the cross-sections will all be squares, as illustrated in the next figure.

\[
square1 := \text{implicitplot3d}(y = 1, x = -\sqrt{7} .. \sqrt{7}, y = 0.5 .. 1.5, z = -\sqrt{7} .. \sqrt{7}, \text{color} = \text{red}) ; \\
square2 := \text{implicitplot3d}(y = 2, x = -2 .. 2, y = 1.5 .. 2.5, z = -2 .. 2, \text{color} = \text{green}) ; \\
square3 := \text{implicitplot3d}(y = 0, x = -\sqrt{8} .. \sqrt{8}, y = -0.5 .. 0.5, z = -\sqrt{8} .. \sqrt{8}, \text{color} = \text{pink}) ; \\
square4 := \text{implicitplot3d}(y = -2, x = -2 .. 2, y = -2.5 .. -1.5, z = -2 .. 2, \text{color} = \text{green}) ; \\
square5 := \text{implicitplot3d}(y = -1, x = -\sqrt{7} .. \sqrt{7}, y = -1.5 .. -0.5, z = -\sqrt{7} .. \sqrt{7}, \text{color} = \text{red}) ; \\
display(\{\text{top, bottom, front, back, curve1, curve2, square1, square2, square3, square4, square5}\}) ;
\]
For each \( y \) for which there is an associated cross-section, the length of the corresponding square is twice the \( x \)-coordinate on the cylinder \( x^2 + y^2 = 8 \) (or twice the \( z \)-coordinate on the cylinder \( y^2 + z^2 = 8 \)), which is

\[
2\sqrt{8 - y^2}.
\]

Therefore, the cross-sectional area is \( A(y) = 4\left(8 - y^2\right) \). The volume is then computed as an integral with respect to \( y \).

\[
\text{Int}\left(4\left(8 - y^2\right), y = -\sqrt{8}..\sqrt{8}\right) = \text{int}\left(4\left(8 - y^2\right), y = -\sqrt{8}..\sqrt{8}\right);
\]

\[
\int_{-\sqrt{8}}^{\sqrt{8}} \left(32 - 4y^2\right) \, dy = \frac{256}{3} \sqrt{2}
\]

(1.2)

**Exercise**

Redo the above visualization and computation with cylinders having a radius other than \( \sqrt{8} \)?
Can you generalize to find the formula for the volume of the solid of intersection for a radius of \( r \)?

**Extra Credit:** Can this method be adapted to find the volume of the solid at the intersection of three cylinders, such as \( x^2 + y^2 = 8 \), \( x^2 + z^2 = 8 \), and \( y^2 + z^2 = 8 \)?

### Solid #2: The Ice-Cream-Cone

**How much ice cream is in your cone?**

Another classic (and tasty!) solid of intersection occurs when we intersect a half-cone with a hemisphere. Plot and inspect the graph below.

Explain why the solid between these two surfaces is called the "ice-cream-cone" (even though the "Sno-Cone" (TM) might be more appropriate!).

```maple
cone1 := plot3d(surd(x^2 + y^2, 2), x=-2..2, y=-2..2, view=-1..1.25, color=brown, scaling = constrained, axes = boxed) :
scoop := plot3d(surd(2 - x^2 - y^2, 2), x=-2..2, y=-2..2, view=1..sqrt(2), color=pink, scaling = constrained, axes = boxed) :
display(cone1, scoop);
```
Be aware that I will use the word "cone" to describe the brown surface, and the phrase "ice cream cone" to describe the solid of intersection.

What is the curve of intersection?

The intersection of the surfaces occurs in a plane. Which plane is it?

Let's look at just the volume between these two surfaces. Use your answers above to convince yourself that the following code will show us only "ice cream" and "cone," with no extraneous "drips":

```
cone1 := plot3d(surd(x^2 + y^2, 2), x=-1..1, y=-sqrt(1-x^2)..sqrt(1-x^2), view=-1..1, color = brown, scaling = constrained, axes = boxed) :
scoop := plot3d(surd(2 - x^2 - y^2, 2), x=-1..1, y=-sqrt(1-x^2)..sqrt(1-x^2), view=1 ..sqrt(2), color = pink, scaling = constrained, axes = boxed) :
display(cone1, scoop);
```
It wasn't difficult to find the curve of intersection, but using a rectangular integral to find volumes might become onerous. Inspect the graph carefully...we're intersecting a sphere and a cone. It seems most logical to use spherical coordinates.

\textbf{I don't believe you. What's so great about spherical coordinates?}

Why are spherical coordinates preferred? Well, as we look down the z-axis, the portion of the cone in each of the four quadrants is symmetric—in cylindrical or spherical coordinates, we could integrate $\theta$ from $0$ to $\frac{\pi}{2}$. Consider some rays originating at the origin:

\begin{verbatim}
cone1 := plot3d(\sqrt{x^2 + y^2}, x = -2 .. 2, y = -2 .. 2, view = -1 .. 2, color = brown, scaling = constrained, axes = boxed, transparency = 0.5) :
scoop := plot3d(\sqrt{2 - x^2 - y^2}, x = -1 .. 1, y = -\sqrt{1 - x^2} .. \sqrt{1 - x^2}, view = 1 .. \sqrt{2}, color = pink, scaling = constrained, axes = boxed) :
ray1 := arrow([0, 0, 0], [0, \sqrt{2}, \sqrt{2}], .2, .4, .1, color = black) :
sugarray := arrow([0, 0, 0], [0, 1, \sqrt{3}], .2, .4, .1, color = green) :
raycharles := arrow([0, 0, 0], [0, sqrt(3), 1], .2, .4, .1, color = red) :
display(cone1, scoop, ray1, sugarray, raycharles);
\end{verbatim}
If the ray makes a nonnegative angle of less than $\pi/4$ with the z-axis, then the ray is "inside" of the ice cream cone; if the angle is greater than $\pi/4$, then the ray lies outside of the cone. This should not be a huge surprise--recall that the "standard" cone has equation $\phi=\frac{\pi}{4}$.

Therefore, if we use spherical coordinates, the space inside of the cone is all points with $0 \leq \rho \leq 2$, $0 \leq \phi \leq \pi/4$, $0 \leq \theta \leq 2\pi$...and a little symmetry will make the volume integral much more pleasant.

What are the appropriate bounds for $\theta$, $\phi$, $\rho$?

**Okay, you can check those bounds:**

Let's determine the volume of ice cream in our "ice cream cone."

$$4 \cdot \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\sqrt{2}} \rho^2 \cdot \sin(\phi) \, d\rho \, d\theta \, d\phi;$$

$$\frac{4}{3} \sqrt{2} \pi - \frac{4}{3} \pi$$

(2.1.2.1)
The surface area of an "ice cream cone"

Suppose we want to dip our ice-cream-cone in a thin layer chocolate crunch coating. How much will we need? If we assume the thickness of the coating is negligible (you might say we're "fudging our numbers"), this question really just asks us to find the surface area of the cone.

How do we find the surface area of the solid described by $z = f(x, y)$?

When we discussed the volume of the ice cream cone, you found the intersection of the "scoop" and the "cone.” Let's use this, along with the equations for our surfaces, to set up the surface area integral. Notice that the necessary partial derivatives are very easy to compute by hand, but we'll practice handing them over to Maple.

Let's have Maple finish the integration in polar coordinates:

$$\int_{c}^{d} \int_{a}^{b} \sqrt{\frac{x^2}{2 - x^2 - y^2} + \frac{y^2}{2 - x^2 - y^2}} + 1 \, dy \, dx$$

$$\int_{c}^{d} \sqrt{\frac{1}{2 - x^2 - y^2}} \, dy \, dx$$

(Hmmm...this looks suspiciously like a good "conversion to polar" candidate. What will our limits of integration be? How will the integrand change?)

Let's have Maple finish the integration in polar coordinates:

$$\int_{c}^{d} \int_{a}^{b} \frac{\sqrt{2}}{\sqrt{2 - r^2}} \cdot r, \, r = 0 \ldots 1 \right), \, \theta = 0 \ldots 2\cdot\text{Pi}$$

$$\int_{c}^{d} \int_{a}^{b} \frac{\sqrt{2}}{\sqrt{2 - r^2}} \cdot r, \, r = 0 \ldots 1 \right), \, \theta = 0 \ldots 2\cdot\text{Pi}$$
Exercise: Ice Cream Nirvana

If we assume the ice-cream cone is full (and not dripping!), we can use calculus to determine the volume of ice cream in various-sized cones.

Carefully answer the following questions, and then set up the triple integration that finds the volume of "ice cream" in your "cone."

Let's suppose you have a cone with vertex angle \( \phi = \varphi \); it intersects a sphere to create an ice-cream cone with slant height \( h \).

1. What sort of coordinates best suit this problem?
2. What is the equation for the "ice-cream sphere" in your coordinate system?
3. What is the equation for the "cone" in your coordinate system?
4. Can you use symmetry to simplify your integral?
5. Set up an integral to find the volume of the ice cream cone; have Maple evaluate it. Your formula should depend on \( h \) and \( \varphi \).
6. For extra credit, you may try to find a formula for the surface area of your ice cream cones.