Visualization of Vectors and Span in R^2 and R^3

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Expanded by Steve Mills, Hugh Sanders and Regina Souza --- Draft 6

```
> restart: with(LinearAlgebra): with(plottools): with(plots):
Warning, the name changecoords has been redefined
Warning, the previous binding of the name arrow has been removed and
it now has an assigned value
```

List of things to be fixed:

1. The numbering of the formulas does not match the numbering of the sections
2. The formulas that are copied and pasted not always adhere to the Maple Input convention (uses Math 2D at times) (See command for plotting w2 multiplied by t (first demo of multiplication by scalars in R^3) - and - cannot be edited (there seemly is some authomatic shift-returns in between symbols)
3. Need an intelligent way of defining the (+,+) grid of the linear combination of two vectors in R^3 (or R^2) using integers from 0 to 5, say. (Messy attempt below.)
4.
5. You name it!

Outline

The basic objectives are:

1) Learn the basic mechanics of entering vectors, and producing linear combinations with either addition or scalar multiplication.

2) Learn to plot a set of vectors in R^2 and R^3.

3) Visualize the effects of multiplication by scalar and of addition of vectors in R^2 and R^3

4) Using a random number generator, see what typical linear combinations of a pair of vectors look like.

5) See the effect of linear transformations on the linear combination of vectors

6) Apply these concepts to understand and visualize the parametric description of a line and a plane in R^3

1. Vectors in R^2 and R^3:

The easiest way to enter a vector in R^2 and R^3 is as a list with angle brackets. In Maple you separate the coordinates with commas for a column vector, and with vertical bars (|) for a row vector. The whole vector is surrounded the with angle brackets (< and >).

$v1 := < 1, 1 >;\ v2 := < 1, 3 >;$ $$v1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ $$v2 := \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad (2.1)$$	$w1 := < 1\vert\ 1\vert\ 1 >;\ w2 := < -1\vert\ 3\vert\ 2 >;$ $$w1 := [1 \quad 1 \quad 1]$$ $$w2 := [-1 \quad 3 \quad 2] \quad (2.2)$$

When vectors are entered this way we use normal mathematics notation to add two vectors or to multiply by a scalar.

> $2 * v1;$ $$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad (2.3)$$ >	> $v1 + v2;$ $$\begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad (2.4)$$ >	> $2 * w1 + 3 * w2;$ $$[-1 \quad 11 \quad 8] \quad (2.5)$$ >

Mentally check the computations Maple is doing.

Notice that vectors need to have the same length before we can add them:

```
> <1,2> + <3,4,5>;
Error, (in rtable/Sum) invalid arguments
```

We can also enter vectors in Maple with the Vector command, which is part of the LinearAlgebra package.

> $a1 := Vector([1, 1]);$ $$a1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (2.6)$$ >	> $a2 := Vector([1, 3]);$ $$a2 := \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad (2.7)$$ >	> $a1 + a2;$ $$\begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad (2.8)$$ >	> $3 * a1;$ $$\begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad (2.9)$$ >

Mentally check the computations Maple is doing.

Exercises:

1.1) Use the last 4 digits of your telephone number to create two vectors u1 and u2 in R^2. Use Maple to compute the linear combination 2*u1 + 3*u2. (Be sure to label answers to all exercises. You can either add a comment like "The answer is ..." to the Maple worksheet, or write a comment on your printout.)

>

2.1) Pick six integers from -10 to 10 (repetitions are allowed) to create two distinct nonzero vectors z1 and z2 in R^3. Use Maple to compute 1.0*z1 + 2.0*z2. Compare this to z1+2*z2.

>

2. Plotting Lists of Points:

We plot points representing vectors with the command pointplot, which is part of the plot package.

> $v1 := <1,\ 1>;\ v2 := <1,\ 3>;$

$$v1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v2 := \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{3.1}$$

>

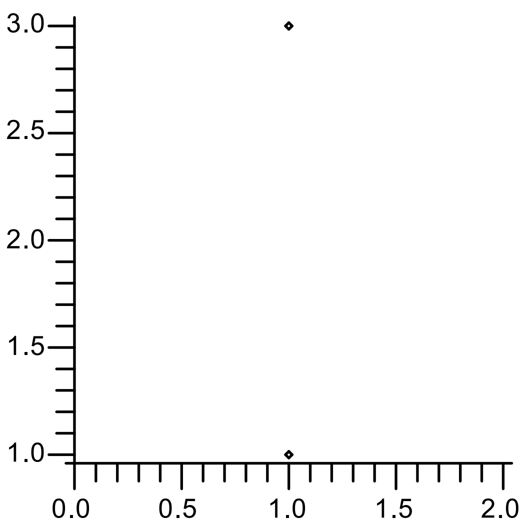
> $a1 := Vector([1, 1]);$
> $a2 := Vector([1, 3]);$

$$a1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$a2 := \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{3.2}$$

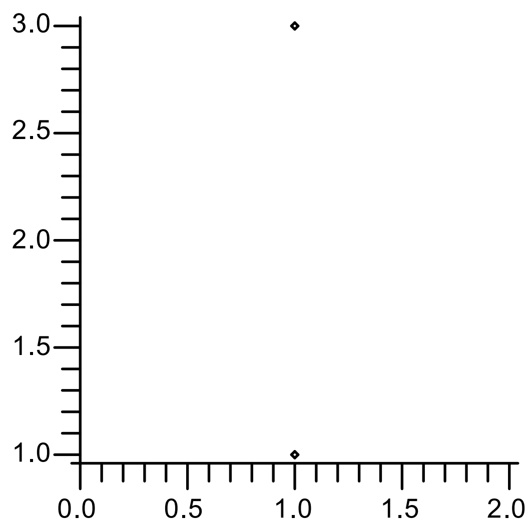
>

> $pointplot(\{v1, v2\});$



>

> $pointplot([a1, a2]);$

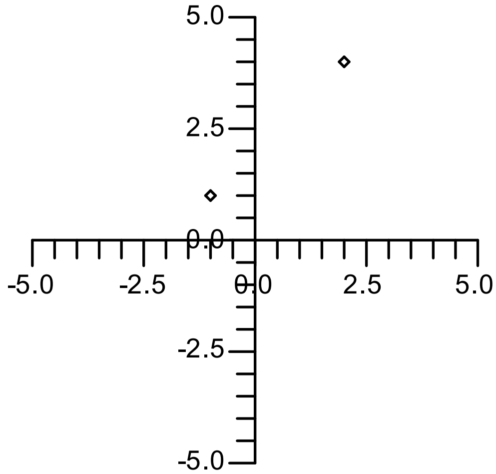


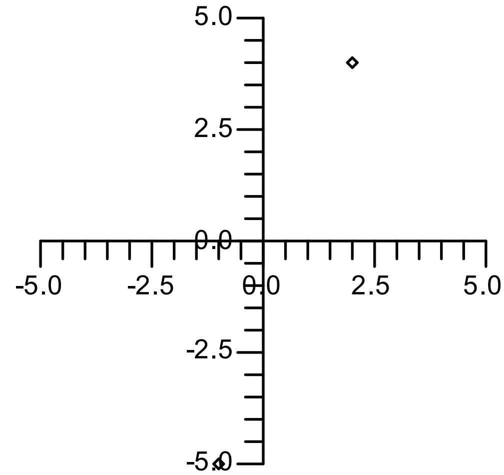
>

Notice that we can plot either a set of points (sets are enclosed in curly braces and are unordered) or a list of points (lists are ordered and enclosed in square brackets). When plotting, you may want to use the view option to specify the viewing window of the plot. For the two plots above, letting x and y both range from -5 to 5 is convenient. You can also specify a symbol size to make the points easier to see.

$pointplot(\{v1 + v2, v2 - 2 * v1\}, view = [-5..5, -5..5],\ symbolsize = 15);$

$pointplot(\{(a1 - 2 * a2), (a1 + a2)\}, view = [-5..5, -5..5],\ symbolsize = 15);$



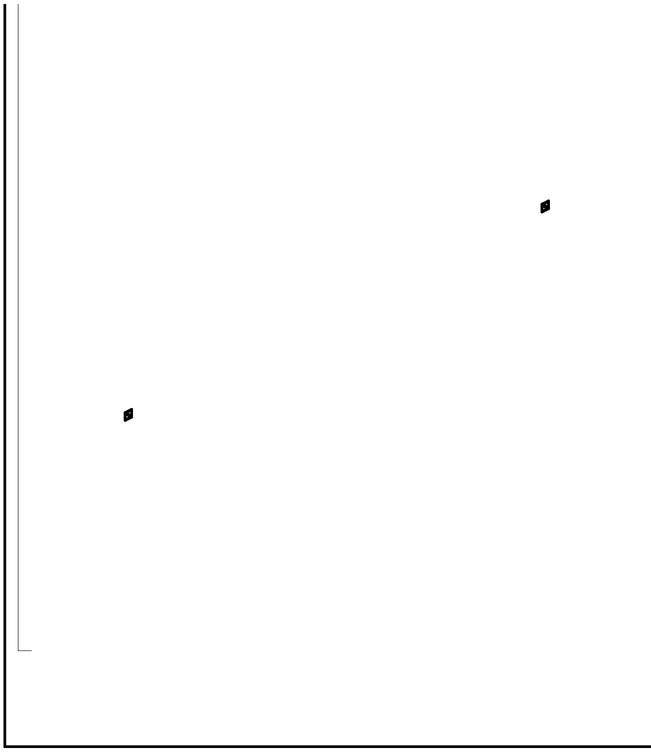


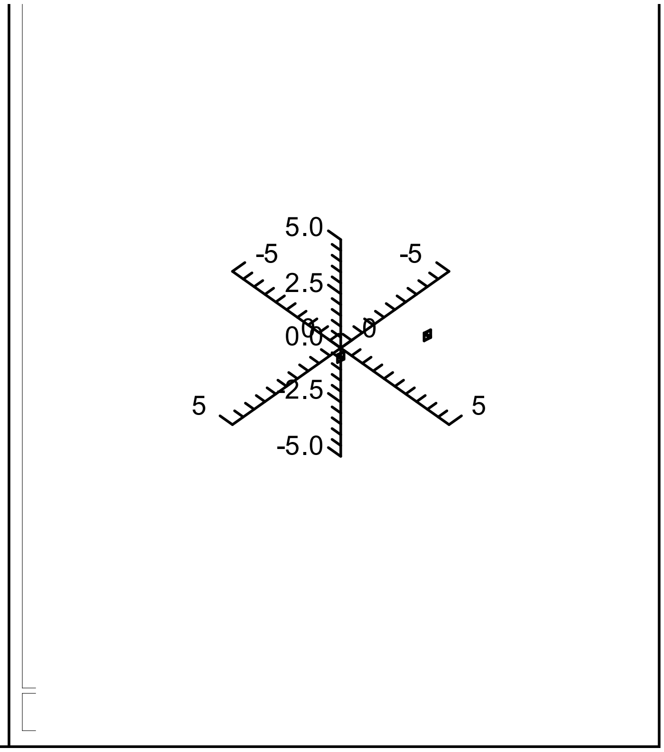
If the vectors are in R^3 instead of R^2, we use the command pointplot3d

Unfortunately, the default option for 3-dimensional plots in Maple is to hide the axes. This can be fixed by either clicking once on the 3-D plot above and then clicking on the icon for normal axes or by using the axes=normal option. Once again there is a view option for these graphs.

```
pointplot3d({w1, w2}
    , color = blue,  symbol = diamond,
    symbolsize = 15 );
```

```
pointplot3d({w1, w2}
    , axes = normal, view = [-5 ..5, -5 ..5,
    -5 ..5 ], color = blue,  symbol =
    diamond,  symbolsize = 15 );
```

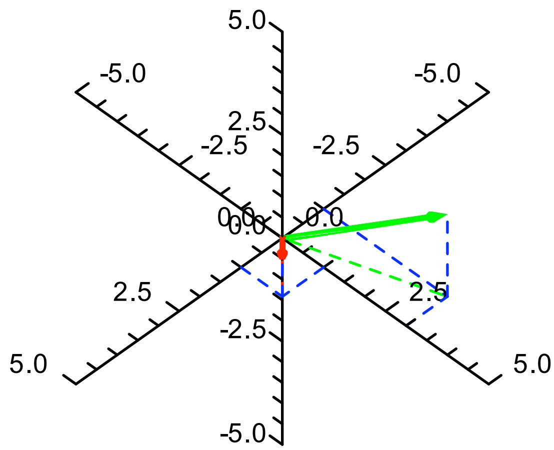




Click on the graph and rotate the plot to get a good idea of the location of the two points.

To help visualize the point in space, it might be helpful to plot the dashed lines (the red and green dashlines indicate the projection of the points on the xy-coordinate plane).

```
> l1 := line([−1, 0, 0],[−1, 3, 0]
    , color = blue, thickness = 1, linestyle = DASH) :
  l2 := line([0, 3, 0],[−1, 3, 0]
  , color = blue, thickness = 1, linestyle = DASH) :
  l3 := line([−1, 3, 0],[−1, 3, 2]
  , color = blue, thickness = 1, linestyle = DASH) :
  l4 := line([0, 0, 0],[−1, 3, 0]
  , color = green, thickness = 1, linestyle = DASH) :
  l5 := arrow([0, 0, 0],[−1, 3, 2],.1, .2, .1, cylindrical_arrow, color = green) :
  l11 := line([1, 0, 0],[1, 1, 0], color = blue, thickness = 1, linestyle = DASH
  ) :
  l12 := line([0, 1, 0],[1, 1, 0]
  , color = blue, thickness = 1, linestyle = DASH) :
  l13 := line([1, 1, 0],[1, 1, 1], color = blue, thickness = 1, linestyle = DASH
  ) :
  l14 := line([0, 0, 0],[1, 1, 0], color = red, thickness = 1, linestyle = DASH
  ) :
  l15 := arrow([0, 0, 0],[1, 1, 1],.1, .2, .1, cylindrical_arrow, color = red) :
  display([l11, l12, l13, l14, l15, l1, l2, l3, l4, l5], axes = normal, view =[
  -5 .. 5, -5 .. 5, -5 .. 5]);
```



We can think of the tip of the red and green segments as the points, and of the arrows as the vectors.

Click on the graph and rotate the plot to get a good idea of the location of the two points.
After you click on the graph, try typing θ=−160 [Enter] and φ=60 [Enter] on the boxes (located at the top left of the tool bar).
Which angles θ and φ give you a good view of these vectors?

Exercises:

2.1) Plot the points [1, 1], [2, -2], [-3, 3], and [4, -4] all on the same graph.

>

2.2) Using the points z1 and z2 you defined in Exercise 2 above, plot z1, z2, z1 + z2, and 2*z1 - z2 all on the same graph.

>

2.3) Include the dashed lines and vectors to the pictures of z1, z2 and z1+z2. Choose angles θ and φ that give you a good view of these three vectors.

>

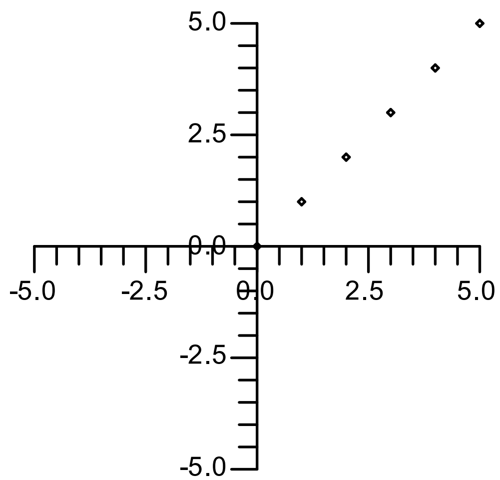
3. Visualizing operations with vectors:

Let us start by visualizing the effect of multiplying vectors by scalars.

Let us use v1=[1 1] defined above.

Using the sequence command, we will graph the multiples of v1 in the window below, using scalars `t` from 0 to 5.

```
> v1MultByt := {seq(t*v1,t=0..5)}:
pointplot(v1MultByt,view=[-5..5,-5..5]);
```




```
>
```

What happens if you use scalars `s` from −5 to 0? Modify the commands and plot the points in the window below.

```
>
```

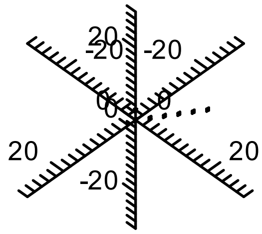
What is the pattern?

Is it any different in R^3? Take a look below :

Using the sequence command, we will graph the multiples of w2=[-1 3 2] in the window below, using scalars `t` from 0 to 5.

$w2MultByt := \{seq(t*w2, t=0..5)\}$:
$pointplot3d(w2MultByt, axes=normal, view = [-30..30, -30..30, -30..30], color=blue, symbol=diamond, symbolsize=5)$;

What happens if you use scalars `s` from −5 to 0? Modify the commands and plot the points in the window below.



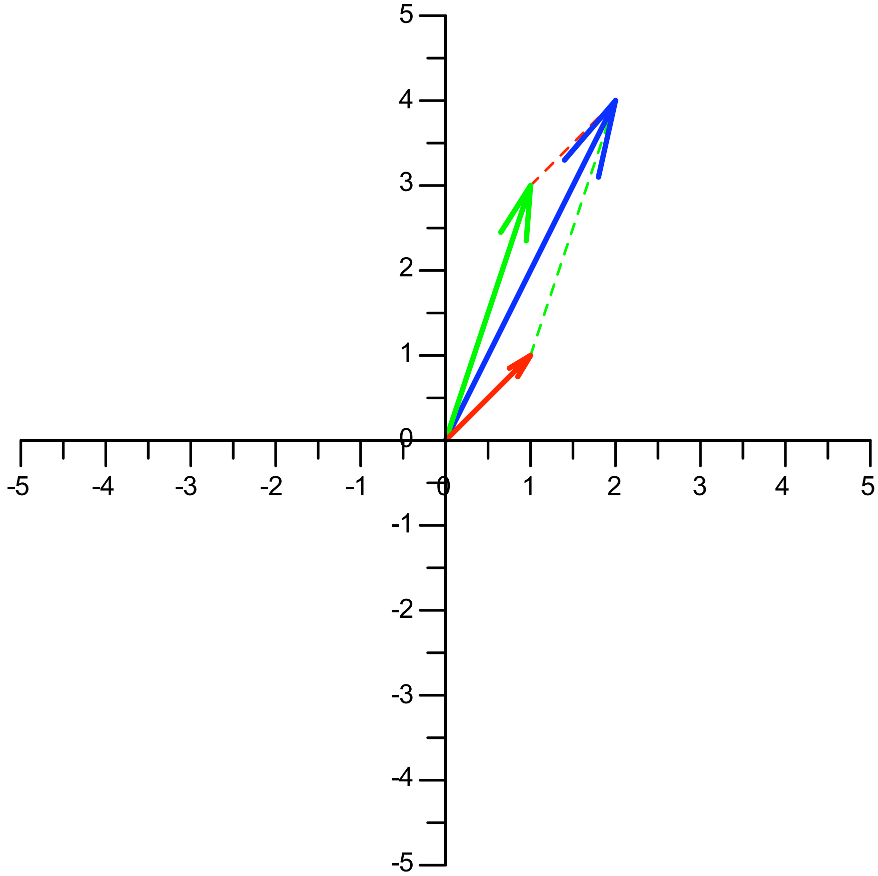
What is the pattern?

Now let us see the geometric properties of addition of vectors.

In will use v1=[1,1] and v2=[1,3] redefined below.

Here we will include the arrows (and segments parallel to them) to make clear what is happening geometrically.

```
> v1 := [1, 1] : v2 := [1, 3] :
vectorv1 := (plots[arrow](v1,color=red, shape=arrow,thickness=2)):
vectorv2 := (plots[arrow](v2,color=green, shape=arrow, thickness=2)):
sumv1v2 := (plots[arrow](v1+v2,color=blue, shape=arrow, thickness=2)):
translv2:= line(v1,v1+v2, color = green, thickness = 1, linestyle = DASH):
translv1:= line(v2,v1+v2, color = red, thickness = 1, linestyle = DASH):
display([vectorv1, vectorv2, sumv1v2,translv1,translv2],view=[-5..5,-5..5]);
```

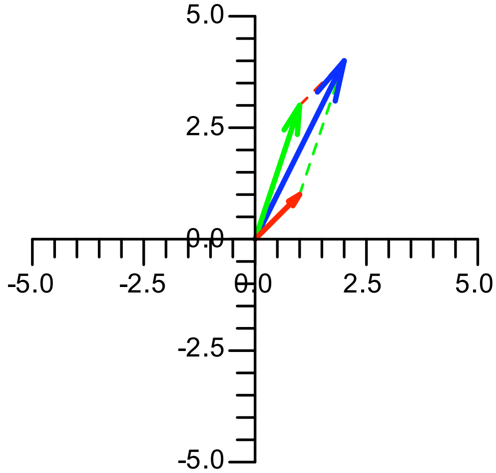


Make the analogous picture for v1-v2 (rename new lines, so you can plot the other plot and the new one side by side

>

Graph the previous picture and your picture side by side

v1+v2 again:	v1-v2
display([*vectorv1*, *vectorv2*, *sumv1v2*, *translv1*, *translv2*], *view*=[-5 ..5, -5 ..5]);	

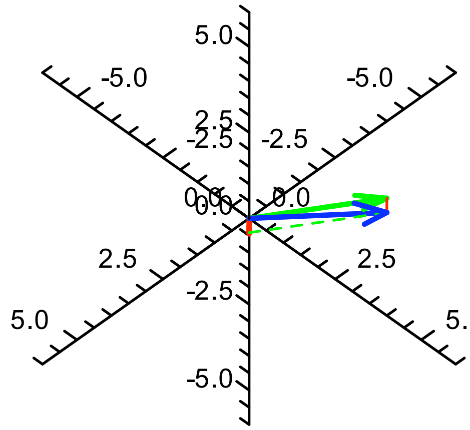


Let us see what happens in R^3:

Let us see what happens in R^3:
We will use w1=[1,1,1] and w2=[-1,3,2] redefined below.

Here we will include the arrows (and segments parallel to them) to make clear what is happening geometrically.

```
> w1 := [1, 1, 1] : w2 := [− 1, 3, 2] :
vectorw1 := (plots[arrow](w1,color=red, shape=arrow,thickness=
2)):
vectorw2 := (plots[arrow](w2,color=green, shape=arrow,
thickness=2)):
sumw1w2 := (plots[arrow](w1+w2,color=blue, shape=arrow,
thickness=2)):
translw2:= line(w1,w1+w2, color = green, thickness = 1,
linestyle = DASH):
translw1:= line(w2,w1+w2, color = red, thickness = 1, linestyle
= DASH):
display([vectorw1, vectorw2, sumw1w2,translw1,translw2],axes=
normal,view=[-6..6,-6..6,-6..6]);
```



Modify the code to get the analogous picture for v1-v2 (rename new lines, so you can plot and compare v1+v2 with v1-v2).

>

Now, in the same way we played with the scalars when trying to visualize the effect of multiplication of a vector by a scalar, we are going to vary the scalars and then add the vectors as in w = s*w1+t*w2. The vector w is said to be a linear combination of w1 and w2.

> $w1w2grid := plot3d(s*w1 + t*w2, s = -5..5, t = -5..5, color = red, transparency = .6, axes = normal, style = patchnogrid);$

```
tw1 := {seq(t*w1,t=0..5)}:
tw2 := {seq(t*w2,t=0..5)}:
tw1w2 := {seq(t*w1+w2,t=0..5)}:
tw12w2 := {seq(t*w1+2*w2,t=0..5)}:
tw13w2 := {seq(t*w1+3*w2,t=0..5)}:
tw14w2 := {seq(t*w1+4*w2,t=0..5)}:
tw15w2 := {seq(t*w1+5*w2,t=0..5)}:
tw1grid := pointplot3d(tw1, axes=normal,view = [-30..30, -30.
.30, -30..30],color=red, symbol=diamond, symbolsize=5):
```

```
tw2grid := pointplot3d(tw2, axes=normal,view = [-30..30, -30.
.30, -30..30],color=green, symbol=diamond, symbolsize=5):
tw1w2grid := pointplot3d(tw1w2, axes=normal,view = [-30..30,
-30..30, -30..30],color=violet, symbol=diamond, symbolsize=5):
tw12w2grid := pointplot3d(tw12w2, axes=normal,view = [-30..30,
-30..30, -30..30],color=orange, symbol=diamond, symbolsize=5):
tw13w2grid := pointplot3d(tw13w2, axes=normal,view = [-30..30,
-30..30, -30..30],color=pink, symbol=diamond, symbolsize=5):
tw14w2grid := pointplot3d(tw14w2, axes=normal,view = [-30..30,
-30..30, -30..30],color=brown, symbol=diamond, symbolsize=5):
tw15w2grid := pointplot3d(tw15w2, axes=normal,view = [-30..30,
-30..30, -30..30],color=coral, symbol=diamond, symbolsize=5):
display([vectorw1, vectorw2, sumw1w2,translw1,translw2,w1grid,
w2grid,w1w2grid,tw1w2grid,tw12w2grid,tw13w2grid,tw13w2grid,
tw14w2grid,tw15w2grid],axes=normal,view=[-1..10,-1..10,-1..10]);
```

```
Warning, unable to evaluate the function to numeric values in the
region; see the plotting command's help page to ensure the calling
sequence is correct
```

$$w1w2grid := INTERFACE\_PLOT3D(\ldots)$$

```
Error, (in display) first argument must be a plot structure or
list, set or array of plot structures
```

Remarks:

In R3:

>

>

Remarks:

Exercises:

3.1) Let vector v1=[-5 4]. Choose a scalar α such that the vector v2=α*v1 is twice as long as v1. Choose a scalar β such that v3=β*v1 has the same length as v1 but it is on a different quadrant. Verify the claims about the lengths by using Pythagoras theorem. Plot the vectors v1, v2 and v3 using colorful arrows.

>

3.2) Addition of vectors in R2.

>

3.3) Multiplication by scalars in R3.

>

3.4) Addition of vectors in R3.

>

4. Random Number Generators and Long Lists:(nothing new)

5. The effect of Linear Transformations on Linear

Combinations (Steve mailed that part - Mike has given some further suggestions)

6. Parametric Equations for Lines and Planes:

In this part of the worksheet we are going to look at parametric equations for lines and planes in R^3. First, for parametric equations of a line, let's enter a vector **v** in R^3 that will be in the direction of the line and a vector $\mathbf{r_0}$ in R^3 that will be pointing from the origin to a point on the line.
So, first enter the vector **v**:

(Here we'll have the student enter a vector, such as, **v** = <1, 4, -2>)

```
> v := <1, 4, -2>;
```

$$v := \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \quad (7.1)$$

(We will define a to be the x component of **v**, b to be the y component of **v** and c to be the z component of **v**.
So, in this example a = 1, b = 4, and c = -2)

```
> a := v[1];
  b := v[2];
  c := v[3];
```

$$a := 1$$
$$b := 4$$
$$c := -2 \quad (7.2)$$

Next, enter the vector $\mathbf{r_0}$:

(Here we'll have the student enter a vector, such as, $\mathbf{r_0}$ = <5, 1, 3>)

```
> r0 := <5, 1, 3>;
```

$$r0 := \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \quad (7.3)$$

(We will define x_0 to be the x component of $\mathbf{r_0}$, y_0 to be the y component of $\mathbf{r_0}$, z_0 to be the z component of $\mathbf{r_0}$
So, in this example $x_0 = 5$, $y_0 = 1$, and $z_0 = 3$.

```
> x0 := r0[1];
  y0 := r0[2];
  z0 := r0[3];
```

$$x0 := 5$$

$$y0 := 1$$

$$z0 := 3 \quad (7.4)$$

The parametric equations for this line are the following:

$x = x_0 + at$

```
> x := x0 + a*t;
```

$$x := 5 + t \quad (7.5)$$

$y = y_0 + bt$

```
> y := y0 + b*t;
```

$$y := 1 + 4\,t \quad (7.6)$$

$z = z_0 + ct)$

```
> z := z0 + c*t;
```

$$z := 3 - 2\,t \quad (7.7)$$

The graph of this line is as follows:

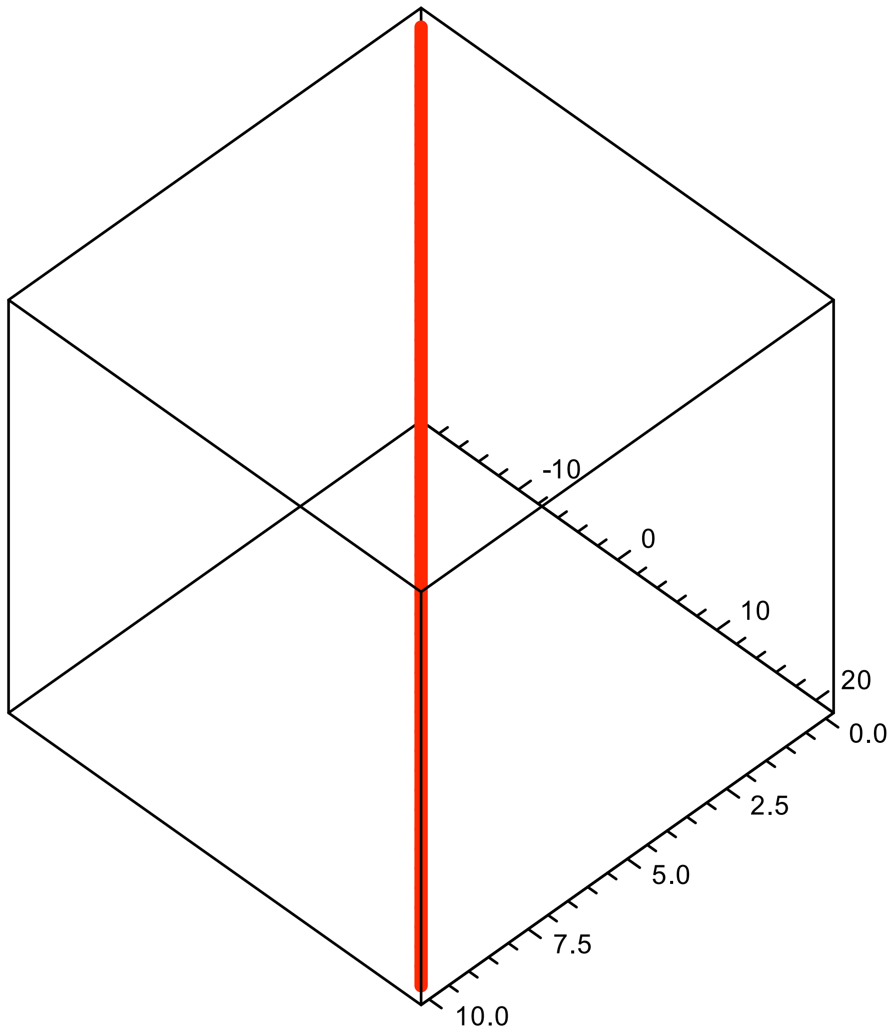
(Graph the vector function:

$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$**)**

```
> r := r0 + t*v;
```

$$r := \begin{bmatrix} 5 + t \\ 1 + 4\,t \\ 3 - 2\,t \end{bmatrix} \quad (7.8)$$

```
> spacecurve(r,t=-5..5,axes=boxed, thickness=5,color=red);
```



>

(Here would be the Maple code for plotting this line.)

Now, let's look at the equation of a plane in R^3. Let's use the two vectors that we used to obtain the parametric equations of the line. So, for our plane, lets let the vector $\mathbf{r_0}$ point from the origin to a point on the plane and the vector **r** be a vector that is normal to the plane.

The equation of the plane that meets these conditions is the following:

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

(For our example, the equation would be $1(x - 5) + 4(y - 1) -2(z - 3) = 0$)

(Graph this function.)

(Here would be the Maple code for plotting this plane.)

Exercise:

6.1)

>

6.2)

>