Orthogonal Matrices and Rotations in R^3

Worksheet by Frank Rooney

```
> restart: with(LinearAlgebra): with(plots): with(plottools):
Warning, the name changecoords has been redefined
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Outline

The objectives are:

1) How to find the rotation in R^3 represented by a given proper orthogonal matrix.
2) How to find the orthogonal matrix that represents a rotation in R^3

>

Rotations in R^3 corresponding to a given orthogonal matrix

An orthogonal matrix Q satisfies
$Q.Q^T = Q^T.Q = I$
The vector corresponding to the axis of rotation is left unchanged by the matrix in other words it is an eigenvector of Q with eigenvector 1.
Firstly let's generate a random orthogonal matrix

```
> va := RandomVector(3):
  vb := RandomVector(3):
  vec1 := Normalize(va,Euclidean):
  vc := vb - (vec1.vb)*vec1:
  vec2 := Normalize(vc,Euclidean):
  vec3 := CrossProduct(vec1,vec2):
  Q := <vec1 | vec2 | vec3>;
```

$$Q := \Bigg[[\frac{46}{2319}\sqrt{1546}, \frac{58507}{13020320013}\sqrt{8680213342}, -\frac{3307}{26040640026}\sqrt{1546}\sqrt{8680213342}], [-\frac{31}{4638}\sqrt{1546}, \frac{249791}{26040640026}\sqrt{8680213342}, \frac{2585}{26040640026}\sqrt{1546}\sqrt{8680213342}], [\frac{67}{4638}\sqrt{1546}, \qquad (2.1)$$

$$-\frac{45101}{26040640026}\sqrt{8680213342},\ \frac{5737}{26040640026}\sqrt{1546}\sqrt{8680213342}]$$

>

Just to check.

>

```
> QT := Transpose(Q):
  QT.Q; Q.QT;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.2)$$

Now the axis of rotation is the eigenvector corresponding to the eigenvalue 1.

```
> EigQ := Eigenvectors (Q):
  evalf(%);
```

$$\begin{bmatrix} 0.7403466090 + 0.6722253326\,I \\ 0.7403466090 - 0.6722253326\,I \\ 1. \end{bmatrix}, \begin{bmatrix} [-0.2663632913 + 1.034203589\,I, -0.2663632913 - 1.034203589\,I, 0.7704195769], \\ [-0.5242027802 - 0.5255101304\,I, -0.5242027802 + 0.5255101304\,I, 1.516185218], [1., 1., 1.] \end{bmatrix} \qquad (2.3)$$

> The axis is the eigenvector corresponding to 1.(Note the complex eigenvalues are of the form cos θ ± i sin θ, with θ the rotation angle)

```
> A := DeleteColumn(EigQ[2],1..2):
  B := <A[1,1],A[2,1],A[3,1]>:
  Normalize(B,Euclidean):
  axis1 := evalf(%);
```

$$axis1 := \begin{bmatrix} 0.3904995968 \\ 0.7685029483 \\ 0.5068661397 \end{bmatrix} \quad (2.4)$$

> There is another way to find the axis of rotation. If w is the eigenvector of Q corresponding to 1 then
> $Q.w = w = Q^T.w$ so $(Q - Q^T).w = 0$
> or if q is the axial vector corresponding to $Q - Q^T$ this means q x w = 0 or w is parallel to q

```
> Qsk := Q - QT:
  q := < Qsk[2,3],Qsk[3,1],Qsk[1,2]>:
   Normalize(q,Euclidean):
  axis := evalf(%);
```

$$axis := \begin{bmatrix} 0.3904995967 \\ 0.7685029478 \\ 0.5068661393 \end{bmatrix} \quad (2.5)$$

>

> To find the angle of rotation we rely on the invariance of the trace. A rotation about the z axis has the form

$$Rot = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There is some orthogonal matrix P that rotates the z axis to the axis of rotation for Q.
Then $Q = P.\,Rot\,.\,P^T$
Trace (Q) = Trace $(P.\,Rot\,.\,P^T)$ =
Trace $(P^T.P.\,Rot)$ = Trace (Rot) = $1 + 2\cos(\theta)$

```
> trQ := Trace(Q) :
  arccos((trQ-1)/2):
  theta = evalf(%);
```

$$\theta = 0.7372105004 \quad (2.6)$$

>

Finding the matrix corresponding to a given rotation in R^3

The rotation matrix will depend only on the axis **n** and the angle of rotation α so it has to have the form
$\mathbf{Q} = A\,\mathbf{I} + B\,\mathbf{n} \otimes \mathbf{n} + C\,\mathbf{N}$

where **N** is the skew symmetic matrix with **n** as its axial vector and A, B, and C are functions of α. Using the facts that **Q.n = n**, **Trace (Q) =** 1 + 2 cos α, and **Q . Q**^{T}= **I**, we calculate that A = cos α, B = 1 - cos α and C = sin α
First get a random axis and random angle of rotation.

```
> vec := RandomVector(3):
  n := Normalize(vec,Euclidean):
  nbyn := << n[1]*n[1],n[2]*n[1],n[3]*n[1]>|<n[1]*n[2],n[2]*n[2],n[3]*n[2]>|<n[1]*n[3],n[2]*n[3],n[3]*n[3]>>:
   N := << 0,n[3],-n[2]>|<-n[3],0,n[1]>|<n[2],-n[1],0>>:
  alpha := Pi/6:
  Qrot := cos(alpha) * IdentityMatrix(3) + (1-cos(alpha))*nbyn+sin(alpha)*N;
```

$$Qrot := \left[\left[\frac{113}{468}\sqrt{3} + \frac{121}{234}, \frac{77}{234} - \frac{77}{468}\sqrt{3} - \frac{2}{39}\sqrt{26}, \frac{44}{117} - \frac{22}{117}\sqrt{3} + \frac{7}{156}\sqrt{26}\right], \left[\frac{77}{234} - \frac{77}{468}\sqrt{3} + \frac{2}{39}\sqrt{26}, \frac{185}{468}\sqrt{3} + \frac{49}{234}, \frac{28}{117} - \frac{14}{117}\sqrt{3} - \frac{11}{156}\sqrt{26}\right], \left[\frac{44}{117} - \frac{22}{117}\sqrt{3} - \frac{7}{156}\sqrt{26}, \frac{28}{117} - \frac{14}{117}\sqrt{3} + \frac{11}{156}\sqrt{26}, \frac{85}{234}\sqrt{3} + \frac{32}{117}\right] \right] \tag{3.1}$$

```
>
```

Check this has the correct properties.

```
>   vec; Qrot.vec; arccos((Trace(Qrot)-1)/2);
   Qrot . Transpose(Qrot);simplify(%);
   Transpose(Qrot).Qrot;simplify(%);
```

$$\begin{bmatrix} 22 \\ 14 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 22 \\ 14 \\ 16 \end{bmatrix}$$

$$\frac{1}{6}\pi$$

[

$$\left(\frac{113}{468}\sqrt{3}+\frac{121}{234}\right)^2+\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}-\frac{2}{39}\sqrt{26}\right)^2 + \left(\frac{44}{117}-\frac{22}{117}\sqrt{3}+\frac{7}{156}\sqrt{26}\right)^2,$$

$$\left(\frac{113}{468}\sqrt{3}+\frac{121}{234}\right)\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}+\frac{2}{39}\sqrt{26}\right) + \left(\frac{77}{234}-\frac{77}{468}\sqrt{3}-\frac{2}{39}\sqrt{26}\right)\left(\frac{185}{468}\sqrt{3}+\frac{49}{234}\right)+\left(\frac{44}{117}-\frac{22}{117}\sqrt{3} + \frac{7}{156}\sqrt{26}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}-\frac{11}{156}\sqrt{26}\right),$$

$$\left(\frac{113}{468}\sqrt{3}+\frac{121}{234}\right)\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}-\frac{7}{156}\sqrt{26}\right) + \left(\frac{77}{234}-\frac{77}{468}\sqrt{3}-\frac{2}{39}\sqrt{26}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}+\frac{11}{156}\sqrt{26}\right) + \left(\frac{44}{117}-\frac{22}{117}\sqrt{3}+\frac{7}{156}\sqrt{26}\right)\left(\frac{85}{234}\sqrt{3}+\frac{32}{117}\right)],\ [$$

$$\left(\frac{113}{468}\sqrt{3}+\frac{121}{234}\right)\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}+\frac{2}{39}\sqrt{26}\right) + \left(\frac{77}{234}-\frac{77}{468}\sqrt{3}-\frac{2}{39}\sqrt{26}\right)\left(\frac{185}{468}\sqrt{3}+\frac{49}{234}\right)+\left(\frac{44}{117}-\frac{22}{117}\sqrt{3} + \frac{7}{156}\sqrt{26}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}-\frac{11}{156}\sqrt{26}\right),$$

$$\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}+\frac{2}{39}\sqrt{26}\right)^2+\left(\frac{185}{468}\sqrt{3}+\frac{49}{234}\right)^2 + \left(\frac{28}{117}-\frac{14}{117}\sqrt{3}-\frac{11}{156}\sqrt{26}\right)^2,$$

$$\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}+\frac{2}{39}\sqrt{26}\right)\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}-\frac{7}{156}\sqrt{26}\right)+\left(\frac{185}{468}\sqrt{3} + \frac{49}{234}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}+\frac{11}{156}\sqrt{26}\right) + \left(\frac{28}{117}-\frac{14}{117}\sqrt{3}-\frac{11}{156}\sqrt{26}\right)\left(\frac{85}{234}\sqrt{3}+\frac{32}{117}\right)],\ [$$

$$\left(\frac{113}{468}\sqrt{3}+\frac{121}{234}\right)\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}-\frac{7}{156}\sqrt{26}\right)$$
$$+\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}-\frac{2}{39}\sqrt{26}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}+\frac{11}{156}\sqrt{26}\right)$$
$$+\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}+\frac{7}{156}\sqrt{26}\right)\left(\frac{85}{234}\sqrt{3}+\frac{32}{117}\right),$$
$$\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}+\frac{2}{39}\sqrt{26}\right)\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}-\frac{7}{156}\sqrt{26}\right)+\left(\frac{185}{468}\sqrt{3}\right.$$
$$\left.+\frac{49}{234}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}+\frac{11}{156}\sqrt{26}\right)$$
$$+\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}-\frac{11}{156}\sqrt{26}\right)\left(\frac{85}{234}\sqrt{3}+\frac{32}{117}\right),$$
$$\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}-\frac{7}{156}\sqrt{26}\right)^2+\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}+\frac{11}{156}\sqrt{26}\right)^2$$
$$\left.+\left(\frac{85}{234}\sqrt{3}+\frac{32}{117}\right)^2]\right]$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Big[\,[$$

$$\left(\frac{113}{468}\sqrt{3}+\frac{121}{234}\right)^2+\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}+\frac{2}{39}\sqrt{26}\right)^2$$
$$+\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}-\frac{7}{156}\sqrt{26}\right)^2,$$
$$\left(\frac{113}{468}\sqrt{3}+\frac{121}{234}\right)\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}-\frac{2}{39}\sqrt{26}\right)+\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}\right.$$
$$\left.+\frac{2}{39}\sqrt{26}\right)\left(\frac{185}{468}\sqrt{3}+\frac{49}{234}\right)$$

$$+\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}-\frac{7}{156}\sqrt{26}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}+\frac{11}{156}\sqrt{26}\right),$$

$$\left(\frac{113}{468}\sqrt{3}+\frac{121}{234}\right)\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}+\frac{7}{156}\sqrt{26}\right)+\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}\right.$$

$$\left.+\frac{2}{39}\sqrt{26}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}-\frac{11}{156}\sqrt{26}\right)$$

$$+\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}-\frac{7}{156}\sqrt{26}\right)\left(\frac{85}{234}\sqrt{3}+\frac{32}{117}\right)], [$$

$$\left(\frac{113}{468}\sqrt{3}+\frac{121}{234}\right)\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}-\frac{2}{39}\sqrt{26}\right)+\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}\right.$$

$$\left.+\frac{2}{39}\sqrt{26}\right)\left(\frac{185}{468}\sqrt{3}+\frac{49}{234}\right)$$

$$+\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}-\frac{7}{156}\sqrt{26}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}+\frac{11}{156}\sqrt{26}\right),$$

$$\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}-\frac{2}{39}\sqrt{26}\right)^2+\left(\frac{185}{468}\sqrt{3}+\frac{49}{234}\right)^2$$

$$+\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}+\frac{11}{156}\sqrt{26}\right)^2,$$

$$\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}-\frac{2}{39}\sqrt{26}\right)\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}+\frac{7}{156}\sqrt{26}\right)+\left(\frac{185}{468}\sqrt{3}\right.$$

$$\left.+\frac{49}{234}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}-\frac{11}{156}\sqrt{26}\right)+\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}\right.$$

$$\left.+\frac{11}{156}\sqrt{26}\right)\left(\frac{85}{234}\sqrt{3}+\frac{32}{117}\right)], [$$

$$\left(\frac{113}{468}\sqrt{3}+\frac{121}{234}\right)\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}+\frac{7}{156}\sqrt{26}\right)+\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}\right.$$

$$\left.+\frac{2}{39}\sqrt{26}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}-\frac{11}{156}\sqrt{26}\right)$$

$$+\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}-\frac{7}{156}\sqrt{26}\right)\left(\frac{85}{234}\sqrt{3}+\frac{32}{117}\right),$$

$$\left(\frac{77}{234}-\frac{77}{468}\sqrt{3}-\frac{2}{39}\sqrt{26}\right)\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}+\frac{7}{156}\sqrt{26}\right)+\left(\frac{185}{468}\sqrt{3}\right.$$

$$\left.+\frac{49}{234}\right)\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}-\frac{11}{156}\sqrt{26}\right)+\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}\right.$$

$$\left.+\frac{11}{156}\sqrt{26}\right)\left(\frac{85}{234}\sqrt{3}+\frac{32}{117}\right),$$

$$\left(\frac{44}{117}-\frac{22}{117}\sqrt{3}+\frac{7}{156}\sqrt{26}\right)^2+\left(\frac{28}{117}-\frac{14}{117}\sqrt{3}-\frac{11}{156}\sqrt{26}\right)^2$$

$$+\left(\frac{85}{234}\sqrt{3}+\frac{32}{117}\right)^2]\Bigg]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

>