DRAFT

A (Very) Brief Introduction to Frames in R^2 and R^3

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```
> restart: with(LinearAlgebra):
```

Definition and examples

Let S = {v1, v2, ... } be a sequence (either finite or infinite) of vectors in either R^2 or R^3. S is a ***frame***
if there are finite positive constants A and B so that

A * Norm(x, Euclidean)^2 < = DotProduct(v1, x)^2 + DotProduct(v2, x)^2 + ... <= B * Norm(x, Euclidean)^2

holds for every vector x in the space. The optimal constants (largest A, smallest B) are called the *frame bounds.*
If A = B, the frame is said to be *tight*.

Examples

1. Let S = {v1, v2}, where $v1 := \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ and $v2 := \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$. Let x:= <a,b> be an arbitrary
vector in R^2.

```
> v1:=<sqrt(2)/2, sqrt(2)/2>;
  v2:=<-sqrt(2)/2, sqrt(2)/2>;
  x:=<a,b>;
```

$$v1 := \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$v2 := \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$x := \begin{bmatrix} a \\ b \end{bmatrix} \qquad (1.1.1)$$

Since:

```
> FirstPart := Norm(x,Euclidean)^2;
```

$$FirstPart := |a|^2 + |b|^2 \tag{1.1.2}$$

and since:

```
> SecondPart :=(v1.x)^2 + (v2.x)^2;
```

$$SecondPart := \left(\frac{1}{2}\sqrt{2}\ a + \frac{1}{2}\sqrt{2}\ b\right)^2 + \left(-\frac{1}{2}\sqrt{2}\ a + \frac{1}{2}\sqrt{2}\ b\right)^2 \tag{1.1.3}$$

```
> simplify(SecondPart);
```

$$a^2 + b^2 \tag{1.1.4}$$

It follows that S is a tight frame for R^2 with frame bound of 1.

2. Let S = {v1, v2, v3, v4}, where $v1 := \langle 1, 0\rangle$, $v2 := \langle 0, 1\rangle$, $v3 := \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$, and $v4 := \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$.

Let $x := \langle a, b\rangle$ be an arbitrary vector in R^2.

```
> restart: with(LinearAlgebra):
  v1:=<1,0>; v2:=<0,1>; v3:=<sqrt(2)/2, sqrt(2)/2>; v4:=<-sqrt
  (2)/2, sqrt(2)/2>; x:=<a,b>;
```

$$v1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$v2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$v3 := \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$v4 := \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$x := \begin{bmatrix} a \\ b \end{bmatrix} \tag{1.1.5}$$

Since:

```
> Norm(x,Euclidean)^2;
```

$$|a|^2 + |b|^2 \tag{1.1.6}$$

and since:

```
> (v1.x)^2 + (v2.x)^2 + (v3.x)^2 + (v4.x)^2;
```

$$a^2 + b^2 + \left(\frac{1}{2}\sqrt{2}\,a + \frac{1}{2}\sqrt{2}\,b\right)^2 + \left(-\frac{1}{2}\sqrt{2}\,a + \frac{1}{2}\sqrt{2}\,b\right)^2 \qquad (1.1.7)$$

```
> simplify(%);
```

$$2a^2 + 2b^2 \qquad (1.1.8)$$

it follows that S is a tight frame for R^2 with frame bound equal to 2.

3. Let S = { v1, v2, v3}, where $v1 := \langle 1, 0\rangle$, $v2 := \langle 0, 1\rangle$, and $v3 := \langle 1, 2\rangle$. S is a frame for R^2, but is not tight. Its lower frame bound is 1 and its upper frame bound is 6. We won't calculate the frame bounds
directly (more on this in a minute), but we will verify that there are two vectors x1 and x2 for which

(v1.x1)^2 + (v2.x1)^2 + (v3.x1)^2 = Norm(x1,Euclidean)^2 and
(v1.x2)^2 + (v2.x2)^2 + (v3.x2)^2 = 6 Norm(x2, Euclidean)^2.

Let $x1 := \langle -2, 1\rangle$ and let $x2 := \langle 1, 2\rangle$

```
> restart: with(LinearAlgebra):
  v1:=<1,0>:v2:=<0,1>:v3:=<1,2>:
  x1:=<-2,1>:x2:=<1,2>:
```

First calculate the norms of the two vectors x1 and x2:

```
> Norm(x1, Euclidean)^2;
  Norm(x2, Euclidean)^2;
```

$$5$$

$$5 \qquad (1.1.9)$$

Then tally up the squares of the inner products:

```
> (v1.x1)^2+(v2.x1)^2+(v3.x1)^2;
```

$$5 \qquad (1.1.10)$$

```
> (v1.x2)^2+(v2.x2)^2+(v3.x2)^2;
```

$$30 \qquad (1.1.11)$$

x1 and x2 have the desired properties.

▼ *Matrix Technique for finding frame bounds*

The following lemma is well known, and provides us with a simple technique for finding upper and lower frame bounds for a set of vectors.

Lemma: Let S = {v1, v2, ..., vN} be a finite sequence of vectors in either R^2 or R^3, with N greater than the dimension of the space under consideration. Let M = [v1, v2, ..., vN]
be the matrix representation for S (each vector in S becoming a column in M). Let F = MM*. If the
smallest eigenvalue of F is positive, then S is a frame. Moreover its frame bounds are equal to the values

of the smallest and largest eigenvalues of F.

Let's reconsider the sequence we studied in item (3), above.

```
> restart: with(LinearAlgebra):
  v1:=<1,0>:v2:=<0,1>:v3:=<1,2>:
> M:=<v1| v2| v3>;
```

$$M := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{1.1.1.1}$$

```
> F:=M.Transpose(M);
```

$$F := \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \tag{1.1.1.2}$$

```
> Eigenvalues(F);
```

$$\begin{bmatrix} 6 \\ 1 \end{bmatrix} \tag{1.1.1.3}$$

The lemma tells us that these vectors form a frame for R^2 with frame bounds of 1 and 6.

Here's another example:

```
> restart: with(LinearAlgebra):
  v1:=<1,0,1>: v2:=<1,1,1>: v3:=<3,-1,3>: v4:=<4,0,4>:
> M:=<v1|v2|v3|v4>;
```

$$M := \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 3 & 4 \end{bmatrix} \tag{1.1.1.4}$$

```
> F:=M.Transpose(M);
```

$$F := \begin{bmatrix} 27 & -2 & 27 \\ -2 & 2 & -2 \\ 27 & -2 & 27 \end{bmatrix} \tag{1.1.1.5}$$

```
> Eigenvalues(F);
```

$$\begin{bmatrix} 0 \\ 28 + 6\sqrt{19} \\ 28 - 6\sqrt{19} \end{bmatrix} \tag{1.1.1.6}$$

Since the smallest eigenvalue of F is zero, this set of vectors can not be a frame for R^3.

Reconstruction formula

Let S = {v1, v2, ...} be a tight frame for R^2 or R^3. It can be shown that x = 1/A * (DotProduct(x,v1) v1 + DotProduct(x, v2) v2 + ...)

This equation is called the *reconstruction formula* for x in terms of the vectors in the frame.

Here's a short exercise in which we construct a tight frame, and then verify the reconstruction formula.

```
> restart: with(LinearAlgebra):
  c:=sqrt(2)/7;
  a:=-1/2*c+1/2*sqrt(6-3*c^2);
  b:=-1/2*c-1/2*sqrt(6-3*c^2);
```

$$c := \frac{1}{7}\sqrt{2}$$

$$a := \frac{11}{14}\sqrt{2}$$

$$b := -\frac{13}{14}\sqrt{2} \qquad (2.1)$$

Use these values to construct a tight frame for R^2.

```
> v1:=<a,-1>; v2:=<b,-1>; v3:=<c,-1>;
```

$$v1 := \begin{bmatrix} \frac{11}{14}\sqrt{2} \\ -1 \end{bmatrix}$$

$$v2 := \begin{bmatrix} -\frac{13}{14}\sqrt{2} \\ -1 \end{bmatrix}$$

$$v3 := \begin{bmatrix} \frac{1}{7}\sqrt{2} \\ -1 \end{bmatrix} \qquad (2.2)$$

```
> M:=<v1| v2| v3>;
```

$$M := \begin{bmatrix} \frac{11}{14}\sqrt{2} & -\frac{13}{14}\sqrt{2} & \frac{1}{7}\sqrt{2} \\ -1 & -1 & -1 \end{bmatrix} \qquad (2.3)$$

```
> F:=M.Transpose(M);
```

$$F := \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad (2.4)$$

This shows that the vectors we've constructed form a tight frame with frame bound 3.

Now let's try the reconstruction formula.

```
> x:=<u,v>;
```

(2.5)

$$x := \begin{bmatrix} u \\ v \end{bmatrix} \tag{2.5}$$

```
> 1/3*((x.v1)*v1 + (x.v2)*v2 + (x.v3)*v3);
```

$$\begin{bmatrix} \frac{11}{42}\left(\frac{11}{14}\,\overline{u}\,\sqrt{2}-\overline{v}\right)\sqrt{2}-\frac{13}{42}\left(-\frac{13}{14}\,\overline{u}\,\sqrt{2}-\overline{v}\right)\sqrt{2}+\frac{1}{21}\left(\frac{1}{7}\,\overline{u}\,\sqrt{2}-\overline{v}\right)\sqrt{2} \\ \overline{v} \end{bmatrix} \tag{2.6}$$

```
> simplify(%);
```

$$\begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix} \tag{2.7}$$

```
> 
> 
> 
```

Since we're in a real vector space, this is identical to the vector x.