An Application to ODE's

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Adapted from Chapter 7 of *Elementary Differential Equations and Boundary Value Problems,* Boyce and DiPrima, 8th edition

```
> restart:with(plots):with(LinearAlgebra):
Warning, the name changecoords has been redefined
```

Notes in Green are added for MAA Prep Workshop Participants and not part of the worksheet.

This is a first draft of a notebook that I plan to use in my differential equations course next year. Most, if not all, students taking this course will have had a linear algebra, but they will need a quick review of matrix notation, eigenvalues, and eigenvectors.

▶ Linear Algebra Review

The next four sections present the material to the students. There are no Maple commands here.

▶ Homogeneous Linear Systems with Constant Coefficients

▶ Equilibrium Solutions

▶ Phase Plane

▶ Solving Homogeneous Systems

The next section is the first to utilize Maple.

▶ Using Maple to compute eigenvalues and eigenvectors

▶ Example 1

Consider the following system of first-order ODE's:

$x_1'(t) = x_1(t) + x_2(t)$ *and* $x_2'(t) = 4x_1(t) + x_2(t)$

This system can be written in matrix form as folows:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} \quad \text{or } \mathbf{x}' = \mathbf{A}\mathbf{x}, \text{ where } \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

In Maple we have

```
> A:=Matrix(2,2,[[1,1],[4,1]]);
```

$$A := \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \quad (1)$$

```
> Eigenvectors(A);
```

(2)

$$\left[\begin{array}{c} -1 \\ 3 \end{array}\right], \left[\begin{array}{cc} -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{array}\right] \quad (2)$$

```
> evA := Eigenvectors(A, output='list'):
  evA[1][1]; evA[2][1];
  [evA[1][3][1], evA[2][3][1]];
```

$$-1$$

$$3$$

$$\left[\left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right], \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]\right] \quad (3)$$

The eigenvalues are -1 and 3. Since any constant multiple of an eigenvector is also an eigenvector, we could also write the eigenvector corresponding to the eigenvalue of 3 as $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ instead of

$$\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

Thus, two solutions of the system of ODE's are

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

Why does the following computation show that these solutions form a fundamental set?

```
> fundamentalset:=Matrix(2,2,[[exp(3*t),2*exp(-t)],[exp(3*t),-2*exp(
  -t)]]):
  simplify(Determinant(fundamentalset));
```

$$-4\,e^{(2\,t)} \quad (4)$$

Answer:

So, the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

We can verify this solution with Maple's dsolve command

```
> odesystem:={D(x1)(t)=x1(t)+x2(t), D(x2)(t)=4*x1(t)+x2(t)};
```

$$odesystem := \{ (D(x2))(t) = 4\, x1(t) + x2(t),\ (D(x1))(t) = x1(t) + x2(t) \} \qquad (5)$$

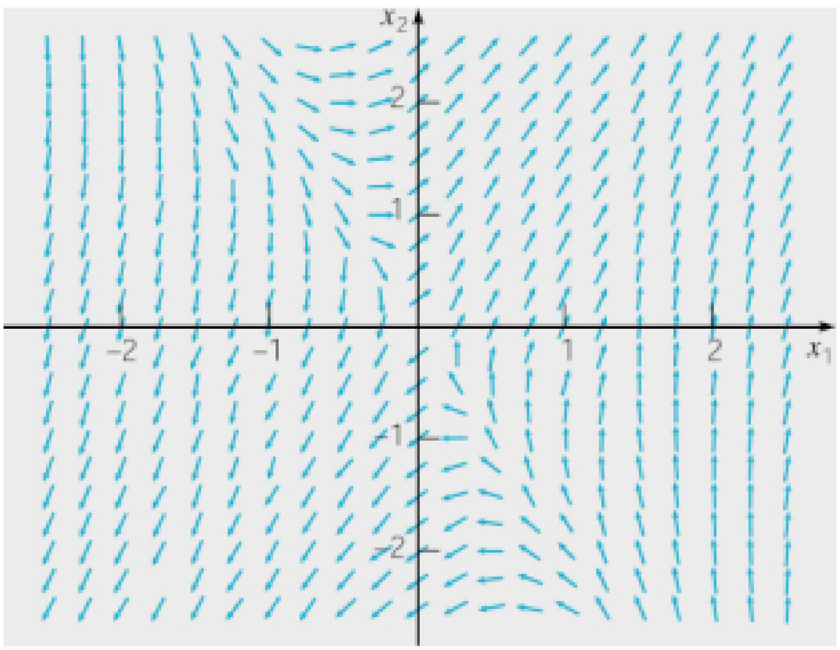
```
> dsolve(odesystem);
```

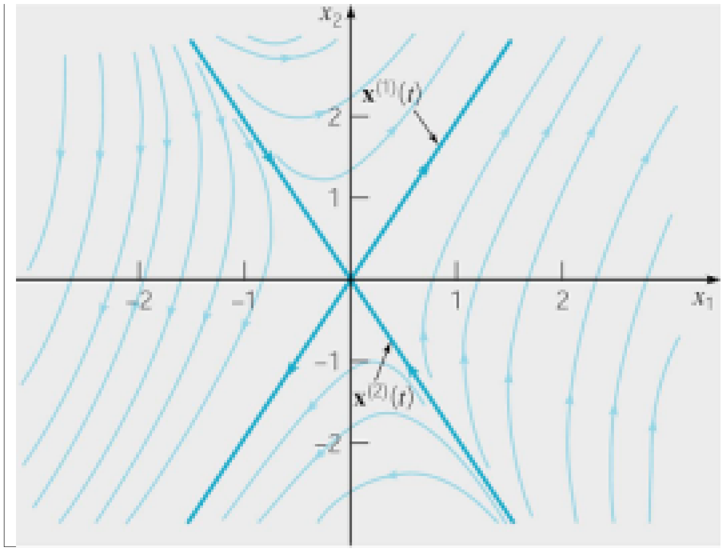
$$\{ x1(t) = \_C1\, e^{(-t)} + \_C2\, e^{(3t)},\ x2(t) = -2\, \_C1\, e^{(-t)} + 2\, \_C2\, e^{(3t)} \} \qquad (6)$$

Now we can graph the phase plane for the general solution

We can note that for nonzero c1 , all solutions asymptotically approach the line x2=2x1 as t approaches infinity. For nonzero c2 all solutions asymptotically approach the line x2=-2x1

To be completed --- I need to reproduce the following graphs using Maple's plot commands





► **Matrix Exponential Functions**

► **Using Maple to compute the Exponential Matrix**

► **Exercises**