Components in the Row Space and Null Space

Worksheet by David A. Huckaby

Hi PREP participants! This is very much a first draft, with little by way of Maple fireworks (the learning of which lies mostly in my future :). Any comments would be much appreciated. [I don't know how to make the R^2; R^2 with the 2 as a superscript. Those that look good (see the second text cell below) were taken from Mike and Russell's worksheets. Also, in the two plots of the codomain, it would be nice to have a y-axis. Also, the colors on the final graph sometimes work so that all three arrows are visible, and sometimes one of the arrows is obscured (without changing the code at all). It's late, so instead of poking around in help, I'll fire this off to our fearless leaders and let you comment if you care to. Thanks!] By the way, there are to be two more parts to the worksheet, as mentioned at the end. It is assumed students have already seen that the row space and null space are orthogonal complements. This worksheet focuses on the components of a vector in each subspace.

```
> restart: with(LinearAlgebra): with(plots):
Warning, the name changecoords has been redefined
```

Outline

The basic objectives are:

1) Review the relationship between the null space and row space of a matrix.

2) Graphically investigate the components of a vector in the null space and row space.

Simple example: Row space and null space are the *x*- and *y*-axes

First construct a 2-by-2 matrix A with a random entry in the upper left and zeros elsewhere.

```
> Vec1 := <1,2>;
```

$$Vec1 := \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (2.1)$$

```
> A := <<1,2>|<2,4>>;
```

$$A := \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad (2.2)$$

Interpret A as a matrix that represents a linear transformation T from R^2 to R^2

What is a basis for the row space of A?

```
> h := NullSpace(A);
```

$$h := \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} \quad (2.3)$$

Confirm that if **w** is in the null space of A, then A**w** = 0. (Note that since this null space is one-dimensional, every element of the null space is just a multiple of **h**.)

```
> w := rand(-100..100)()*h[1];
  Aw := A.w;
```

$$w := \begin{bmatrix} 190 \\ -95 \end{bmatrix}$$

$$Aw := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.4}$$

What is a basis for the row space of A?

```
> p := RowSpace(A);
```

$$p := [[1 \quad 2]] \tag{2.5}$$

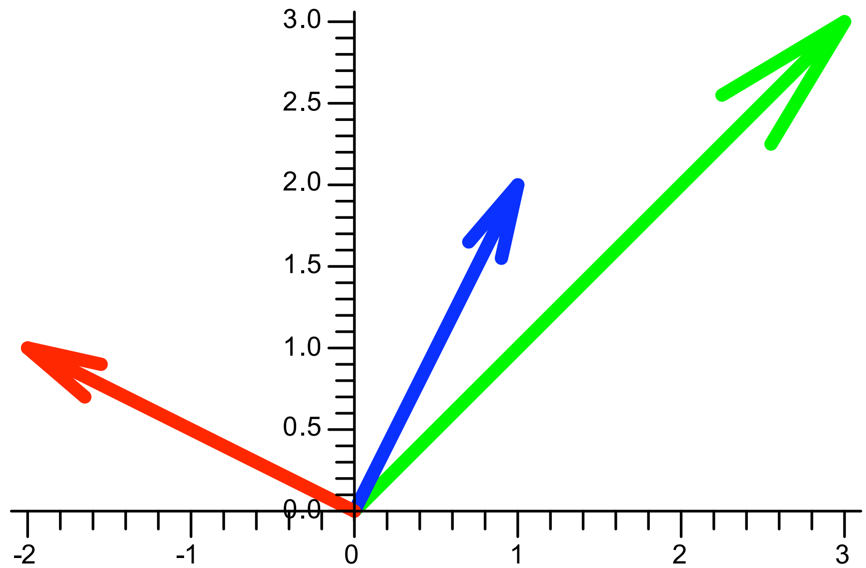
Note that every row of A is a multiple of **p**. Note also that the row space (the *x*-axis) and the null space (the *y*-axis) are perpendicular to each other, as expected.
Now consider an arbitary vector **u** in R^2.

```
> u := Vector([rand(2..4)(),rand(1..4)()]);
```

$$u := \begin{bmatrix} 3 \\ 3 \end{bmatrix} \tag{2.6}$$

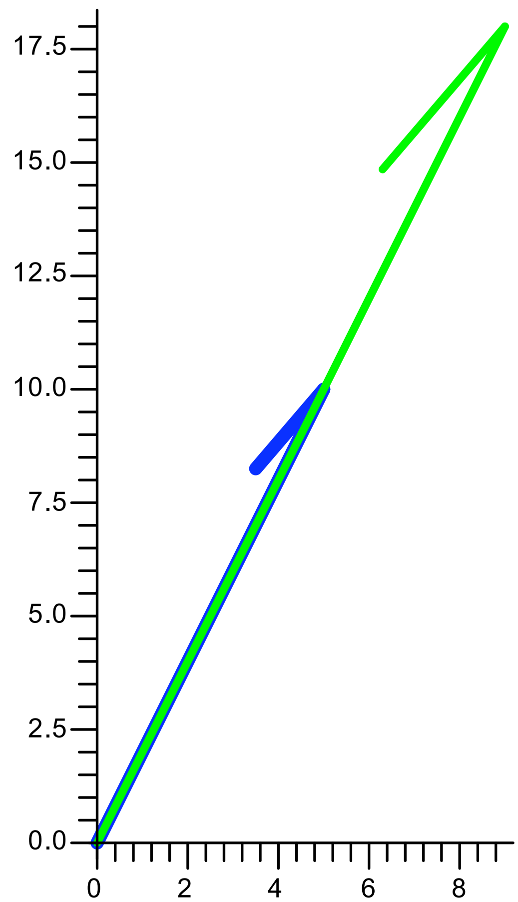
Plot **p**, **h**, and **u** on the same graph.

```
> vp := arrow(p[1],color=blue, shape=arrow,thickness=5):
  vh := arrow(h[1],color=red, shape=arrow,thickness=5):
  vu := arrow([u],color=green, shape=arrow,thickness=5):

  display({vp,vh,vu},scaling=constrained);
```



Now that we know what the domain looks like, let's consider the codomain. Investigate where T maps each of these three vectors.

```
> wp := arrow(A.Transpose(p[1]),color=blue, shape=harpoon,
  thickness=5):
  wh := arrow(A.h[1],color=red, shape=harpoon,thickness=5):
  wu := arrow([A.u],color=green, shape=harpoon,thickness=3):

  display({wp,wh,wu},scaling=constrained);
```



Where did all three vectors get mapped? Where did the red vector **h** get sent? Why? Consider what the column space (i.e., the range) of A is.

```
> colBasis := ColumnSpace(A);
```

$$colBasis := \left[\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right] \qquad (2.7)$$

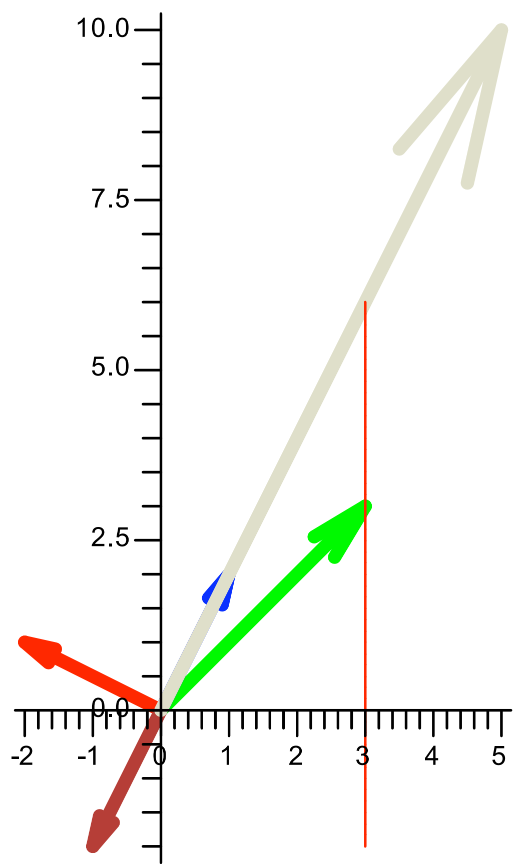
Every linear combination of the columns of A is simply a multiple of this vector, and our graph of the images of **p**, **h**, and **u** above agrees.

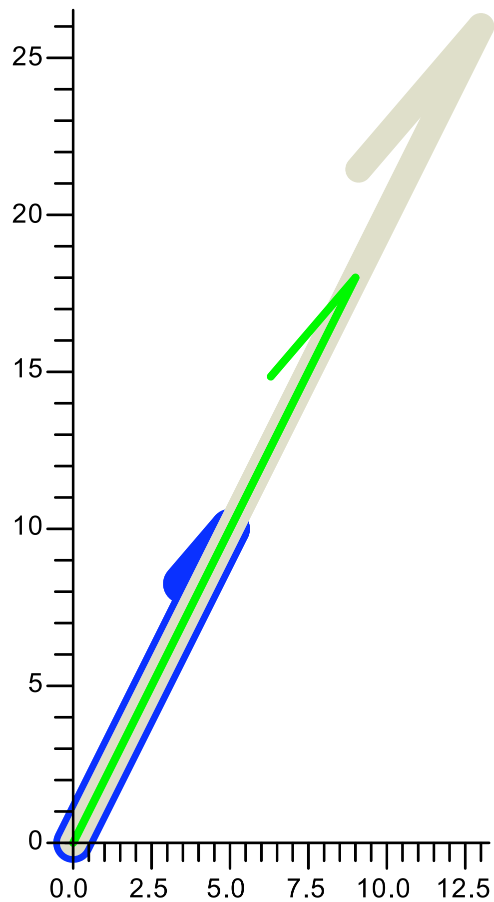
Consider again the domain. The vector **u**, like any other vector in R^2, has a component in the row space of A and a component in the null space of A. It is only the component in the row space that determines where **u** gets mapped. Indeed, if $u = c1p + c2h$;
, then A(c1**p** + c2**h**) = c1A**p** + c2A**h** = c1A**p** + **0** = c1A**p**. Note that c2, that is, the "amount of **u** in the null space," is irrelevant. All that matters is how much of **u** lies in the row space.

Observe the behavior of other vectors that have the same row space component as **u**. First, have another look at the domain. Then observe where all the vectors are mapped.

```
> vu1 := arrow(Vector(5*Vec1),color=wheat, shape=arrow,thickness=
  5):
```

```
vu2 := arrow(Vector(-1*Vec1),color=brown, shape=arrow,
thickness=5):

pl1 := plot([u[1],t,t=-2..6]):

display({vp,vh,vu,vu1,vu2,pl1},scaling=constrained);

wp := arrow(A.Transpose(p[1]),color=blue, shape=harpoon,
thickness=15):
wu1 := arrow(A.Vector([u[1],5]),color=wheat, shape=harpoon,
thickness=10):
wu2 := arrow(A.Vector([u[1],-1]),color=brown, shape=harpoon,
thickness=5):

display([wu,wu1,wu2,wp,wh],scaling=constrained);
```





>

The green, brown, and wheat vectors are all mapped to the same vector in the column space.

Part Two would be to repeat this, but with a row space and null space that are different from the x- and y-axes. (This can really drive home the power of thinking about the components of vectors in certain directions. "Ah! All that matters is how much of **u** is in *that* direction!") Part Three would repeat it once more, this time going from R^3 to R^2.