

Exploring Abstract Algebra with Computer Software

PREP Workshop 2004

Section 1: Working with Permutation Groups

The possible arrangements of 1,2,3,4,5,6 in the puzzle (Figure 1) are the elements in the group generated by $(1,3,4,5,6)$ and $(1,3,2)$. Enter this group into GAP.

```
gap> G:= SymmetricGroup(6);
Sym( [1 .. 6] )
gap> r:=(1,3,4,5,6);
(1,3,4,5,6)
gap> s:=(1,3,2);
(1,3,2)
gap> K:= Subgroup(G, [r,s]);
Group([(1,3,4,5,6), (1,3,2)]);
gap> Elements(K);
```

The elements in the subgroup K describe all the possible arrangements of this puzzle. If you want to know how to get the original arrangement of the puzzle into a particular arrangement, use the `Factorization` command in GAP. For example, to see how to change the original arrangement to the arrangement in Figure 2 type

```
gap> Factorization(K, (2,3,4));
x1^4*x2*x1
```

The `x1` denotes the first generator of K and `x2` denotes the second generator of K . Thus the above output from GAP tells us $(2,3,4) = (1,3,4,5,6)^4 * (1,3,2) * (1,3,4,5,6)$. (Remember that GAP multiplies permutations from left to right.)

Section 1, Project 1:

1.1 Indicate what arrangement of the puzzle in Figure 1 corresponds to the following permutations. Use GAP to determine how (if possible) to get the following arrangements from the arrangement in Figure 1.

a) $(4,5,6)$

- b) (2,3)
- c) (1,2)(3,4)
- d) (1,2)(3,4)(5,6).

Check the answers to parts a and c by hand. The function `Factorization(K,a)` in `GAP` will return `fail` if a cannot be expressed in terms of the generators of K .

1.2 Repeat Exercise 1.1 for the puzzle in Figure 3. Hint: there are two permutations that generate all possible permutations of this puzzle. Check the answer to part d by hand.

The cycle structure of a permutation is the number of 2-cycles, 3-cycles, etc. it contains when it is written as the product of disjoint cycles. For example (1,2,3)(4,5) and (1,3,6)(2,7) have the same cycle structure.

Section 1, Project 2:

1.3 Let $a = (1, 5, 10)$ and $b = (1, 3, 5, 7, 9, 11)$.

- a) What do you think will be the cycle structure of b^2 , b^3 and b^6 ? Check your answer using `GAP`.
- b) Compute ab . What do you think will be the cycle structure of $(ab)^3$ and $(ab)^4$?

The function `CycleStructurePerm(a)` in `GAP` gives the cycle structure of the permutation a . The cycle structure of (1,2,3)(4,5)(6,7) for example, is denoted by [2,1]. (The first spot in the bracket notation denotes the number of 2-cycles, the second spot the number of 3-cycles, etc.) The cycle structure of (1,2,3)(4,5,6,7,8) is denoted by [,1,,1]. This means there are no 2-cycles, one 3-cycle, no 4-cycles and one 5-cycle. The absence of a number after a comma indicates there are no cycles of that length.

```
gap> CycleStructurePerm((1,2,3)(4,5)(6,7));
[ 2, 1 ]
gap> CycleStructurePerm((1,2,3)(4,5,6,7,8));
[ , 1, , 1 ]
```

The following function lists all elements x , in a permutation group G that have the cycle structure s :

```

gap> cstruc:= function(G,s)
> return Filtered(Elements(G), x -> CycleStructurePerm(x) = s);
> end;
function( G, s ) ... end

```

(Thanks to Alexander Hulpke for providing this function.) Type this function into GAP. We can now use this function to, for example, find all the elements in S_6 that have one 2-cycle and one 4-cycle:

```

gap> cstruc(SymmetricGroup(6),[1,,1]);
[ (1,2)(3,4,5,6), (1,2)(3,4,6,5), (1,2)(3,5,6,4), (1,2)(3,5,4,6),
(1,2)(3,6,5,4), (1,2)(3,6,4,5), (1,2,3,4)(5,6), (1,2,3,5)(4,6),
(1,2,3,6)(4,5), (1,2,4,3)(5,6), (1,2,4,6)(3,5), (1,2,4,5)(3,6),
(1,2,5,3)(4,6), (1,2,5,6)(3,4), (1,2,5,4)(3,6), (1,2,6,3)(4,5),
(1,2,6,5)(3,4), (1,2,6,4)(3,5), (1,3,4,2)(5,6), (1,3,5,2)(4,6),
(1,3,6,2)(4,5), (1,3)(2,4,5,6), (1,3)(2,4,6,5), (1,3,2,4)(5,6),
(1,3,5,6)(2,4), (1,3,6,5)(2,4), (1,3)(2,5,6,4), (1,3)(2,5,4,6),
(1,3,2,5)(4,6), (1,3,4,6)(2,5), (1,3,6,4)(2,5), (1,3)(2,6,5,4),
(1,3)(2,6,4,5), (1,3,2,6)(4,5), (1,3,4,5)(2,6), (1,3,5,4)(2,6),
(1,4,3,2)(5,6), (1,4,6,2)(3,5), (1,4,5,2)(3,6), (1,4,2,3)(5,6),
(1,4,5,6)(2,3), (1,4,6,5)(2,3), (1,4)(2,3,5,6), (1,4)(2,3,6,5),
(1,4,6,3)(2,5), (1,4)(2,5,6,3), (1,4)(2,5,3,6), (1,4,2,5)(3,6),
(1,4,3,6)(2,5), (1,4,5,3)(2,6), (1,4)(2,6,5,3), (1,4)(2,6,3,5),
(1,4,2,6)(3,5), (1,4,3,5)(2,6), (1,5,3,2)(4,6), (1,5,6,2)(3,4),
(1,5,4,2)(3,6), (1,5,2,3)(4,6), (1,5,6,4)(2,3), (1,5,4,6)(2,3),
(1,5)(2,3,4,6), (1,5)(2,3,6,4), (1,5,6,3)(2,4), (1,5)(2,4,6,3),
(1,5,2,4)(3,6), (1,5,3,6)(2,4), (1,5)(2,4,3,6), (1,5,4,3)(2,6),
(1,5)(2,6,4,3), (1,5,3,4)(2,6), (1,5)(2,6,3,4), (1,5,2,6)(3,4),
(1,6,3,2)(4,5), (1,6,5,2)(3,4), (1,6,4,2)(3,5), (1,6,2,3)(4,5),
(1,6,5,4)(2,3), (1,6,4,5)(2,3), (1,6)(2,3,4,5), (1,6)(2,3,5,4),
(1,6,5,3)(2,4), (1,6)(2,4,5,3), (1,6,2,4)(3,5), (1,6,3,5)(2,4),
(1,6)(2,4,3,5), (1,6,4,3)(2,5), (1,6)(2,5,4,3), (1,6,3,4)(2,5),
(1,6)(2,5,3,4), (1,6,2,5)(3,4) ]

```

Section 1, Project 2:

1.4 Use GAP to find the number of permutations in S_9 of the following forms:
a) A product of a 4-cycle, and two 2-cycles (for example $(1, 2, 3, 4)(5, 6)(7, 8)$)

- b) A product of a 5-cycle and a 4-cycle
- c) A product of three 3-cycles
- d) A product of four 2-cycles.

1.5 The command `Centralizer(G,a)` finds the centralizer of an element a in a group G . Find the size of centralizer of each of the following elements in S_9 :

- a) $(1, 2, 3, 4)(5, 6)(7, 8)$ and $(5, 1, 3, 4)(2, 6)(7, 8)$
- b) $(1, 2, 3, 4, 5)(6, 7, 8, 9)$ and $(1, 4, 9, 6, 7)(2, 3, 5, 8)$
- c) $(1, 2, 3)(4, 5, 6)(7, 8, 9)$ and $(1, 5, 8)(2, 4, 9)(3, 6, 7)$
- d) $(1, 2)(3, 4)(5, 6)(7, 8)$ and $(1, 9)(2, 8)(3, 7)(4, 6)$.

Based on these answers, for any element $a \in S_n$, make a conjecture about the number of elements in the centralizer of a and the number of element in the centralizer of any permutation in S_n with the same cycle structure as a . Test your conjecture out for some elements of S_7 .

1.6 Find a relationship between the answers you obtained in each part of Exercises 1.4 and 1.5 and the order of S_9 .

1.7 Pick an element in S_9 and call it a . Compare its cycle structure to the cycle structure of the permutation bab^{-1} for

- a) $b = (1, 2, 3, 4, 5, 6, 7, 8, 9)$
- b) $b = (1, 2)(3, 4)(5, 6)(7, 8)$
- c) $b = (1, 2, 3, 4)(5, 6, 7, 8)$.

1.8 Repeat Exercise 1.7 for a different element a in S_9 .

1.9 Make a conjecture about, given two elements a and b in a group of permutations G , how the cycle structure of a and bab^{-1} are related. Test your conjecture for a pair of elements in the dihedral group D_{50} .

1.10 Based on your conjecture in Exercise 1.9, make a conjecture about a relationship between the order of an element a and the order of bab^{-1} .

1.11 The command for finding the order of an element a in GAP is `Order(a)`. Let $a = (1, 2)$. For the elements b in Exercise 1.7 compute the orders of ab and ba . In these three cases is it true that $|ab| = |ba|$?

The elements $r = (1, 3, 4, 5, 6)$ and $s = (1, 3, 2)$ above generated A_6 :

```
gap> Size(Group([(1,3,4,5,6), (1,3,2)]));
360
gap> Factorial(6)/2;
360
```

Using the fact that A_n is the only subgroup of S_n of order $|S_n|/2$, we know that r and s generate A_6 .

In the following exercises we investigate the subgroup of S_n generated by the $(n-1)$ -cycle $(1, 3, 4, 5, \dots, n)$ and the 3-cycle $(1, 3, 2)$ or the 4-cycle $(1, 4, 3, 2)$.

Section 1, Project 4:

1.13 Use **GAP** to help you conjecture what subgroup of S_n is generated by $a = (1, 2)$ and $b = (1, 2, \dots, n)$.

1.14 Using **GAP** for at least eight values of n determine the subgroup of S_n generated by the $(n-1)$ -cycle $(1, 3, 4, 5, \dots, n)$ and the 3-cycle $(1, 3, 2)$.

1.15 Using **GAP** for at least eight values of n determine the subgroup of S_n generated by the $(n-1)$ -cycle $(1, 3, 4, 5, \dots, n)$ and the 4-cycle $(1, 4, 3, 2)$.

1.16 Explain **why** you get different subgroups of S_n in Exercises 5.14 and 5.15 depending on whether n is even or odd and on whether the second generator is a 3 or 4-cycle.

At this point, you may find it interesting to note how **GAP** updates the information about a group (or other constructed object) as it determines characteristics of the group. For example, create the subgroup of S_5 generated by the permutations $(1,3)$ and $(1,4,5)$:

```
gap> g:= Group([(1,3), (1,4,5)]);
Group([ (1,3), (1,4,5) ])
```

The command `KnownAttributesOfObject` returns the current information **GAP** contains on your group:

```
gap> KnownAttributesOfObject(g);
[ "LargestMovedPoint", "GeneratorsOfMagmaWithInverses",
  "MultiplicativeNeutralElement" ]
```

(You can use the the command line help, ? followed by the command, to see what any of these three characteristics mean.) If we now compute with `g` and then reuse the command `KnownAttributesOfObject`, we see that GAP now has more information about our group `g`:

```
gap> Size(g);
24
gap> KnownAttributesOfObject(g);
[ "Size", "OneImmutable", "LargestMovedPoint", "NrMovedPoints",
  "MovedPoints",
  "GeneratorsOfMagmaWithInverses", "MultiplicativeNeutralElement",
  "Pcgs",
  "GeneralizedPcgs", "StabChainMutable", "StabChainOptions" ]
```

In determining the order of our group, GAP went through constructions that determined these 8 new attributes.