Exploring Abstract Algebra with Computer Software
PREP Workshop 2004
Section 18: Orbits and Stabilizers

Let G be a group of permutations of a set S. For each $s \in S$ the stabilizer of s in G is the subgroup of G equal to $\{g \in G \mid g(s) = s\}$. The orbit of s under G is the subset of S equal to $\{g(s) \mid g \in G\}$.

The command `Orbit(G,s)` in GAP will give you the orbit of s under G. The command `Stabilizer(G,s)` creates the subgroup of G that is the stabilizer of s. For example:

```
gap> G:=SymmetricGroup(8);
Sym( [ 1 ..  8 ] )
gap> a:= (1,2,3)(4,5,6);
gap> b:= (7,8);
gap> H:=Subgroup(G,[a,b]);
Group([ (1,2,3)(4,5,6), (7,8) ])
gap> Elements(H);
[ (), (7,8), (1,2,3)(4,5,6), (1,2,3)(4,5,6)(7,8), (1,3,2)(4,6,5),
(1,3,2)(4,6,5)(7,8) ]
gap> Orbit(H,1);
[ 1, 3, 2 ]
gap> Orbit(H,7);
[ 7, 8 ]
gap> Stabilizer(H,1);
Group([ (7,8) ])
gap> Stabilizer(H,7);
Group([ (1,2,3)(4,5,6) ])
gap> Elements(Stabilizer(H,7));
[ (), (1,2,3)(4,5,6), (1,3,2)(4,6,5) ]
```

Careful: Notice that the command `Stabilizer(G,s)` returns a statement describing the stabilizer of s in G in terms of the generators of this stabilizer. To see all the elements in this group you need to use the commands `Elements(Stabilizer(G,s))`.

Section 18, Project

18.1 Find the number of elements in $\text{Orbit}(G, s)$ for $G = D_{10}$ and $s = 1, 2, 3$ and 4.

18.2 Find the number of elements in $\text{Stabilizer}(G, s)$ for $G = D_{10}$ and $s = 1, 2, 3$ and 4.

18.3 Repeat Exercises 18.1 and 18.2 for D_{49} and D_{50}.

18.4 Make a conjecture about the number elements in $\text{Stabilizer}(G, s)$ and in $\text{Orbit}(G, s)$ for any $s \in \{1, 2, 3, \ldots, n\}$.

18.5 Explain, geometrically, why your conjecture in Exercise 18.4 is true.

18.6 Generalize the conjecture made in Exercise 18.4 to other finite permutation groups. Use GAP to help you formulate this conjecture.

The conjecture we are hoping the students will discover here is the Orbit-Stabilizer Theorem: Let G be a finite group of permutations of a set S. Then, for any $s \in S$, $|G| = |\text{Stabilizer}(G, s)|\ |\text{Orbit}(G, s)|$.