Exploring Abstract Algebra with Computer Software
PREP Workshop 2004
Section 17: Solvable Groups

GAP has commands for determining when a group G is solvable and, in the case when G is solvable, for producing a series

$$\{e\} = H_0 \subset H_1 \subset \cdots \subset H_k = G$$

such that H_i is normal in H_{i+1} and H_{i+1}/H_i is Abelian for $0 \leq i < k$.

```
gap> S:=SymmetricGroup(3);
Sym( [ 1 .. 3 ] )
gap> DerivedSeries(S);
[ Group([ (2,3), (1,3,2) ]), Group([ (1,3,2) ]), Group(()) ]
```

The above output is a series of subgroups $H_0 = \{e\}$, $H_1 =< (1,3,2) >= \{e, (1,3,2), (1,2,3)\}$ and $S_3 = H_k =< (2,3), (1,3,2) >$. This series shows S_3 is solvable. For another example, the following finds a series for D_4 which shows it is solvable:

```
gap> d4:=Group((1,2,3,4),(1,4)(2,3));
Group([ (1,2,3,4), (1,4)(2,3) ])
gap> DerivedSeries(d4);
[ Group([ (2,4), (1,2,3,4), (1,3)(2,4) ]),
  Group([ (1,3)(2,4) ]), Group(()) ]
```

If the command `DerivedSeries(G)` is used on a group that is not solvable the last element in the series will not be the identity subgroup.

```
gap> S5:=SymmetricGroup(5);
Sym( [ 1 .. 5 ] )
gap> DerivedSeries(S5);
[ Sym( [ 1 .. 5 ] ), Group([ (1,3,2), (2,4,3), (3,5,4) ]) ]
gap> IsSolvable(S5);
false
```

Section 17, Project

17.1 By hand find a series of subgroups of D_n that shows D_n is solvable for $n = 5, 10, 30$.

17.2 Rework Exercise 17.1 using GAP. Is there only one possible such series for a given dihedral group?

17.3 Determine if A_n is solvable for $n = 4$, 5 and 8.

17.4 Determine if $D_4 \oplus D_8$ is solvable.

17.5 Determine if $S_3 \oplus S_3$ is solvable.

17.6 Prove or disprove: The direct product of solvable groups is solvable.