

Exploring Abstract Algebra with Computer Software

PREP Workshop 2004

Section 15: Another False Conjecture

The group of rotations in \mathbf{R}^3 of a 3-prism is isomorphic to D_3 . Let G denote this group of rotations. (See figure of the labeled 3-prism.) The group of rotations must be a subgroup of the group of permutations of the set $\{1, 2, 3, 4, 5, 6\}$. There are two types of rotations. We can rotate each triangle the same amount. Thus $(1, 2, 3)(4, 5, 6)$ is in G . We can also rotate the front facing triangle to the back. Thus the rotation $(1, 4)(2, 6)(3, 5)$ is in G .

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gap> G:=Subgroup(SymmetricGroup(6), [(1,2,3)(4,5,6), (1,4)(2,6)(3,5)]);
Group([ (1,2,3)(4,5,6), (1,4)(2,6)(3,5) ])
gap> Elements(G);
[ (), (1,2,3)(4,5,6), (1,3,2)(4,6,5), (1,4)(2,6)(3,5), (1,5)(2,4)(3,6),
  (1,6)(2,5)(3,4) ]
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The above exhibits G as a subgroup of S_6 . We can now have GAP set up an isomorphism between D_3 and G . (We use here that $D_3 \cong S_3$.)

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gap> d3:= SymmetricGroup(3);
Sym( [ 1 .. 3 ] )
gap> IsomorphismGroups(d3,G);
[ (2,3), (1,2,3) ] -> [ (1,4)(2,6)(3,5), (1,2,3)(4,5,6) ]
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That is, the homomorphism that maps the generators $(2, 3)$ and $(1, 2, 3)$ of D_3 to the generators $(1, 4)(2, 6)(3, 5)$ and $(1, 2, 3)(4, 5, 6)$ respectively of G is an isomorphism.

Section 15, Project

For the following exercises see the accompanying figures for this section.

15.1 Exhibit the group of rotations in \mathbf{R}^3 of a 4-prism as a subgroup of S_4 . This group is isomorphic to which familiar group?

15.2 Exhibit the group of rotations in \mathbf{R}^3 of a 5-prism as a subgroup of S_5 . This group is isomorphic to which familiar group?

15.3 Exhibit the group of rotations in \mathbf{R}^3 of a 6-prism as a subgroup of

S_6 . This group is isomorphic to which familiar group?

15.4 Make a conjecture about what the group of rotations in \mathbf{R}^3 of a n -prism is.

15.5 Prove your conjecture in Exercise 15.4.

15.6 The order of the symmetry group (including both rotations and reflections) in \mathbf{R}^3 of a 3-prism is 12. Exhibit this symmetry group as a subgroup of S_6 . This group is isomorphic to which familiar group of order 12?

15.7 Exhibit the symmetry group in \mathbf{R}^3 of a 5-prism as a subgroup of S_5 . This group is isomorphic to which familiar group?

15.8 Make a conjecture about what the symmetry group in \mathbf{R}^3 of a n -prism is.

15.9 Test your conjecture in Exercise 15.8 for $n = 4$.

(The conjecture most likely made in 15.8 fails in 15.9.)