# Exploring Abstract Algebra with Computer Software
# PREP Workshop 2004
# Section 11: Vector Spaces

Recall $\mathbf{Z}_p$ is a field for every prime $p$. Let $GF(p)^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbf{Z}_p\}$. Then $GF(p)^n$ is an $n$ dimensional vector space over $\mathbf{Z}_p$. In this section we will investigate subspaces of $GF(p)^n$. For example, consider the vector space $GF(3)^2$:

```
gap> V:=GF(3)^2;
( GF(3)^2 )
gap> Elements(V);
[ [ 0*Z(3), 0*Z(3) ], [ 0*Z(3), Z(3)^0 ], [ 0*Z(3), Z(3) ],
[ Z(3)^0, 0*Z(3) ], [ Z(3)^0, Z(3)^0 ], [ Z(3)^0, Z(3) ],
[ Z(3), 0*Z(3) ], [ Z(3), Z(3)^0 ], [ Z(3), Z(3) ] ]
```

(Recall a generator of the multiplicative group of units in $\mathbf{Z}_p$ is denoted in GAP by $Z(p)$.)

*Careful:* Note that $GF(p)^n$ is the direct product of $n$ copies of $\mathbf{Z}_p$ not the field of order $p^n$.

The vector space $V$ is small enough that we can easily find all the 1-dimensional subspaces by hand. The `Display` command is useful here:

```
gap> Display(Elements(V));
. .
. 1
. 2
1 .
1 1
1 2
2 .
2 1
2 2
```

The zero in denoted by a dot and `Z(3)` is denoted by 2. The following GAP work shows the 1-dimensional subspaces:

```
gap> D:=Subspaces(V,1);
Subspaces( ( GF(3)^2 ), 1 )
gap> e:= Elements(D);;
gap> Display(Elements(e));
[ VectorSpace( GF(3), [ [ 0*Z(3), Z(3)^0 ] ] ),
  VectorSpace( GF(3), [ [ Z(3)^0, 0*Z(3) ] ] ),
  VectorSpace( GF(3), [ [ Z(3)^0, Z(3)^0 ] ] ),
  VectorSpace( GF(3), [ [ Z(3)^0, Z(3) ] ] ) ]
gap> Display(Elements(e[1]));
 . .
 . 1
 . 2
gap> Display(Elements(e[2]));
 . .
 1 .
 2 .
gap> Display(Elements(e[3]));
 . .
 1 1
 2 2
gap> Display(Elements(e[4]));
 . .
 1 2
 2 1
```

Thus, as expected, we see that the 1-dimensional subspaces are $\{(0,0),(0,1),(0,2)\}$, $\{(0,0),(1,0),(2,0)\}$, $\{(0,0),(1,1),(2,2)\}$, and $\{(0,0),(1,2),(2,1)\}$. If you just need to find the number of 1-dimensional subspaces you can type:

```
gap> Size(Subspaces(V,1));
4
```

**Section 11, Project**

11.1 **By hand** find all the 1-dimensional subspaces of $GF(3)^3$.

11.2 Use GAP to check your answer to Exercise 11.1.

11.3 Use GAP to find the number of 1-dimensional subspaces of $GF(p)^3$ for $p = 2, 5, 7$ and 11.

11.4 Make a conjecture about the number of 1-dimensional subspaces of $GF(p)^3$. Prove your conjecture.

11.5 **By hand** find all the 2-dimensional subspaces of $GF(3)^3$.

11.6 Use GAP to check your answer to Exercise 11.5.

11.7 Use GAP to find the number of 2-dimensional subspaces of $GF(p)^3$ for $p = 5, 7$ and 11.

11.8 Make a conjecture about the number of 2-dimensional subspaces of $GF(p)^3$.