A "proof" that A_5 is simple

[These problems were assigned just prior to the class period in which we proved that A_5 is simple. By #2, a normal subgroup of A_5 would have to be the union of conjugacy classes, and by #3, one of those would have order 1. The order of the subgroup must divide 60, but #1 shows that the orders of the conjugacy classes make this impossible.]

Problems 1-4 are a warm-up for the proof that A_5 is simple.

1. Suppose you have disjoint sets A, B, C, D, and E, with cardinalities 1, 15, 20, 12, and 12, respectively. H is a set that is formed by taking the union of A with one or more of the sets B, C, D, and E.

   a. List all the possibilities for the cardinality of H.

   b. Which of your answers to part (a) divide 60?

2. Let G be a group, H a normal subgroup of G, and h an element of H. Prove that the conjugacy class of h is a subset of H.

3. In any group G, which is the cardinality of the conjugacy class of the identity?

4. (Bonus) Here are some computations in GAP. Think about how they might relate to problem 2.

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gap> a5:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> Size(a5);
60
gap> ConjugacyClasses(a5);
[ ()^G, (1,2)(3,4)^G, (1,2,3)^G, (1,2,3,4,5)^G, (1,2,3,5,4)^G ]
gap> cc2:=ConjugacyClass(a5, (1,2)(3,4));
(1,2)(3,4)^G
gap> Size(cc2);
15
gap> cc3:=ConjugacyClass(a5,(1,2,3));
(1,2,3)^G
gap> Size(cc3);
20
gap> cc4:=ConjugacyClass(a5,(1,2,3,4,5));
(1,2,3,4,5)^G
gap> Size(cc4);
12
gap> cc5:=ConjugacyClass(a5, (1,2,3,5,4));
(1,2,3,5,4)^G
gap> Size(cc5);
12
```