Assignment on Conjugacy Classes and Sylow Subgroups

[These problems were due ten days after the class period that was spent in the computer classroom, using GAP to find conjugacy classes and Sylow subgroups [See handout on "Conjugacy Classes and Sylow Subgroups."] The assignment included the generic statement that the students could use GAP, if they wished, but the assignment included other problems for which GAP was irrelevant. The goal of the GAP problems was to help students to explore a particular example of a group, to use GAP to reinforce the content of theorems about orders of conjugacy classes, numbers of Sylow subgroups, etc., and to encourage them to test conjectures and find counterexamples.]

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1. Pick an interesting group whose order is more than 50. (At a minimum, "interesting" means that the order of the group is divisible by at least two distinct primes.) Call your group G.
   a. Find all the conjugacy classes of G and list their elements.
   b. Verify that the sum of the orders of the conjugacy classes equals the order of G.
   c. Verify that the order of each conjugacy class divides the order of G.
   d. Verify that the order of each conjugacy class equals the order of G, divided by the order of the appropriate subgroup of G.
   e. What group would be even more "interesting" than the one you chose?

2. Find all the Sylow p-subgroups of the group that you chose in #1. Which ones are normal?

3. Decide whether each of the following statements is true or false. If the statement is true, give a brief proof. If the statement is false, give a counterexample.

   a. If G is a finite group, then the number of one-element conjugacy classes divides the order of G.
   b. If G is a non-abelian finite group, then the number of multi-element conjugacy classes divides the order of G.
   c. If H is a subgroup of G, then the number of distinct conjugates of H in G is equal to $|G| / |N_G(H)|$.