Centralizers, Normalizers, and Centers with GAP

[This is the handout for a class period that was spent in the computer classroom, introducing the students to GAP, demonstrating some of its features, and giving them the opportunity to use it on their own.] It assumes that the students are familiar with permuation groups, the dihedral group with eight elements, and the definitions of center, centralizer, normal subgroup, and normalizer. They also need a working knowledge of what it means for a group to generated by a set of elements. It includes a non-essential reference to Cayley's theorem that can be omitted, if the students are not yet familiar with that theorem. The handout includes some embedded exercises, as well as exercises for the students to work on at the end.]

Centralizers, Normalizers, and Centers with GAP

GAP is a computer algebra system that is specifically designed for computations that involve groups. It works on both the "element level" (finding products of elements, for example) and on the "group level" (answering questions about groups and subgroups). It also does ordinary arithmetic and modular arithmetic. There is an on-line version of the GAP manual at

mirrors.ccs.neu.edu/GAP/NEU/Info4/manual.html

In this assignment, you are going to explore some questions about groups, centers of groups, normalizers, centralizers. For the time being, we'll look only at permutation groups. According to Cayley's theorem, every group is isomorphic to a subgroup of a permutation group, so this restriction really doesn't limit the scope of our inquiry.

I. Starting GAP

[This section begins with instructions about how to start GAP in the computer lab.]

The computer starts with the "gap>" prompt.

You will probably want to save your work in a file. To do that, make up a file name and type

LogTo("filename");

<u>Warnings</u>: GAP is case-sensitive, so "LogTo" is not the same as "Logto." Also note that you <u>must</u> end every GAP command with a semicolon.

II. Permutations and permutation groups

GAP "thinks" of permutations as cycles, or products of cycles. The cycle (123) in S_3 is represented as (1,2,3) by GAP. The commas are essential.

To find the product of two permutations, just use "*" to denote the operation of composition. For example, to compute the product of (12) and (23), type

(1,2)*(2,3);

GAP responds with

(1,3,2)

Question: In computing the product of cycles, does GAP start at the left or the right? How might this affect your work?

Symmetric groups. To define the group S_4 in GAP, we use the fact that S_4 is generated by (1234) and (12). Type

s4:=Group((1,2,3,4), (1,2));

Remarks: "s4:="is GAP-speak for "let s4 denote ..." This command defines s4 as the group that is generated by (1234) and (12). Notice the commas and parentheses, and remember that the first letter of "Group" must be capitalized. And don't forget the semicolon!

GAP responds with

Group([(1,2,3,4), (1,2)])

and thus confirms that it has defined s4 as the group generated by the set {(1,2,3,4), (1,2)}.

If you are wondering whether (1234) and (12) really generate all of S_4, you can type

Elements(s4);

GAP will respond by listing all of the elements in s4, listing them as cycles or products of cycles. You can check that every permutation in S_4 is there.

A quicker way to verify that s4 is all of S_4 is to ask GAP to count the elements of s4. To do that, type

Size(s4);

GAP will respond with

24

which tells you that the order of s4 is 24. Since 24 is the order of S_4, you can conclude that s4 equals S_4.

Dihedral groups. Since dihedral groups can be viewed as permutations of the vertices of an n-gon, you can use permutations to define them, too. For example, D_8 is the subgroup of S_4 that is generated by (1234) and (24). In GAP-speak, you can type

d8:=Group((1,2,3,4), (2,4));

To check that d8 really is D_8, you can use the "Elements" or "Size" command, if you wish.
Subgroups. You can make up other subgroups of S_4, too. For example, the command

h:=Group((1,2,3));

defines h as the group that is generated by the permutation (123). It is thus a subgroup of S_4 (in fact, of S_n for all $n \geq 3$).

<u>Note</u>: For technical reasons, when you are naming groups and other objects in GAP, it's safer to choose names that start with lower case, rather than capital, letters.

III. Normalizers and centralizers

GAP will find the normalizer of a subgroup of a group, the centralizer of a subgroup of a group, the centralizer of an element of a group, and the center of a group.

For example, to find the normalizer of h in s4, type

Normalizer(s4,h);

<u>Remark</u>: Note the order in which s4 and h occur in this expression.

GAP responds with

Group([(1,2,3), (2,3)])

which tells you that the normalizer of h is the group generated by (123) and (23). If you want to give the normalizer a name -- say, n -- you should instead use the command

n:=Normalizer(s4,h);

Using the "Size" command, you will discover that n has order 6.

<u>Question</u>: Is h normal in S_4?

Similarly, to find the centralizer of h in s4, type

Centralizer(s4,h);

GAP responds with

Group([(1,2,3)])

which tells you that the centralizer of h in S_4 is the group generated by (123).

If you want to denote that centralizer by c, then you should type

c:=Centralizer(s4,h);

GAP will also find the centralizer of a specific element of a group. For example, to find the centralizer of (1234) in S_4, type

Centralizer(s4, (1,2,3,4));

GAP responds with

Group([(1,2,3,4)])

which tells you that the centralizer of (1234) is the subgroup generated by (1234).

The command "Centre" (Note British spelling!) finds the center of a group. For example, if you type

Centre(s4);

GAP responds with

Group(())

which tells you that the center of S_4 is the trivial group consisting only of the identity. Similarly, the command

Center(d8);

yields the result

Group([(13)(24)])

which tells you that the center of D_8 is the subgroup generated by (13)(24).

IV. Leaving GAP

Before you log off GAP, you will probably want to save your work. The command

LogTo();

tells GAP to stop saving your work in the file that you named at the beginning. To log off GAP, type

quit;

Then you can print your file, if you wish.

V. Remote access to GAP

[This section contains instructions about using GAP from computers that are not in the computer classroom.]

Exercises:

1. Use GAP to do #1 on p. 33, which you already did by hand. Do your answers agree with GAP's?

[This problem involves find the composition of permuations, and decomposing the result into cycles.]

2. Choose a natural number n ($n \geq 4$) and use GAP to explore S_n and D_{2n}. For example, you should choose at least four subgroups of S_n and/or D_{2n} and find their normalizers, centralizers, and centers. Did you choose any subgroups that are normal in S_n and/or D_{2n}?