

## Automorphism Groups

[This handout reinforces the fact that the elements of an automorphism group are functions from the group to itself. It was distributed at a class following the one in which automorphisms and semidirect products were defined. The instructor can make a transparency of the output and use it to translate the output into the notation that represents  $D_8$  as being generated by an element  $r$  of order 4 and an element  $s$  of order two, subject to the relation  $rs=sr^{-1}$ . The complexity of the output is actually an advantage, since it reflects the complexity of the notion of a group whose elements are functions on a group.]

The handout contains two representations of the automorphism group of  $D_8$ . The first treats it as a permutation of  $\{1, 2, 3, 4\}$ . The second representation requires students to be comfortable with the notion of a finitely generated group.]

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Here are two ways that you can use GAP to find  $\text{Aut}(D_8)$ . The first way involves representing  $D_8$  as a subgroup of  $S_4$ .

```
gap> d8:=Group((1,2,3,4), (2,4));
Group([ (1,2,3,4), (2,4) ])
```

```
gap> Size(d8);
8
```

```
gap> Elements(d8);
[ (), (2,4), (1,2)(3,4), (1,2,3,4), (1,3), (1,3)(2,4), (1,4,3,2), (1,4)(2,3) ]
```

```
gap> aud8:=AutomorphismGroup(d8);
<group of size 8 with 3 generators>
```

```
gap> Elements(aud8);
[ IdentityMapping( Group([ (1,2,3,4), (2,4) ] ) ),
```

```
Pcgs([ (2,4), (1,2,3,4), (1,3)(2,4) ]) -> [ (2,4), (1,4,3,2), (1,3)(2,4) ],
```

```
[ (1,2,3,4), (1,3), (2,4) ] -> [ (1,4,3,2), (1,4)(2,3), (1,2)(3,4) ],
```

```
Pcgs([ (2,4), (1,2,3,4), (1,3)(2,4) ]) -> [ (1,2)(3,4), (1,2,3,4), (1,3)(2,4) ],
```

```
[ (1,2,3,4), (1,3), (2,4) ] -> [ (1,4,3,2), (2,4), (1,3) ],
```

[ (1,2,3,4), (1,3), (2,4) ] -> [ (1,2,3,4), (2,4), (1,3) ],

[ (1,2,3,4), (1,3), (2,4) ] -> [ (1,2,3,4), (1,2)(3,4), (1,4)(2,3) ],

[ (1,2,3,4), (1,3), (2,4) ] -> [ (1,4,3,2), (1,2)(3,4), (1,4)(2,3) ] ]

In the second way, you regard  $D_8$  as a finitely generated group with presentation

$D_8 = \langle r, s \mid r^4=1, s^2=1, rs=sr^{-1} \rangle$ . First rewrite the relation  $rs=sr^{-1}$  as  $s^{-1}rsr=1$ .

```
gap> f:=FreeGroup(2);  
<free group on the generators [ f1, f2 ]>
```

```
gap> d8:=f/[f.1^4, f.2^2, f.2^-1*f.1*f.2*f.1] ;  
<fp group on the generators [ f1, f2 ]>
```

```
gap> Size(d8);  
8
```

```
gap> Elements(d8);  
[ <identity ...>, f2, f1^3*f2, f1, f1^2*f2, f1^2, f1^3, f1*f2 ]
```

```
gap> aud8:=AutomorphismGroup(d8);  
<group of size 8 with 3 generators>
```

```
gap> Elements(aud8);  
[ IdentityMapping( <fp group of size 8 on the generators [ f1, f2 ]> ),
```

^f2,

[ f2, f1 ] -> [ f1^3\*f2, f1^3 ],

[ f1, f2 ] -> [ f1^3\*f2\*f1^6\*f2, f1^3\*f2 ],

^f1\*f2,

^f1,

[ f1, f2 ] -> [ f1^3\*f2\*f1^6\*f2, f1^3\*f2\*f1^6 ],

[ f1, f2 ] -> [ f1^3, f1^3\*f2\*f1^6 ] ]

You can use GAP to compute the images of r and s under the inner automorphisms, if you wish:

```
gap> f.1^f.2;  
f2^-1*f1*f2
```

```
gap> f.1^(f.1*f.2);  
f2^-1*f1*f2
```

```
gap> f.2^(f.1*f.2);  
f2^-1*f1^-1*f2*f1*f2
```

```
gap> f.2^f.1;  
f1^-1*f2*f1
```